

NP-completeness of the initial odd-ear prefix

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Abstract

For a finite 2-vertex-connected graph G , let $\beta(G)$ be the largest number of consecutive odd ears at the beginning of an open ear decomposition, counting its initial circuit and its one-edge ears. We prove that deciding $\beta(G) \geq t$, with t part of the input, is NP-complete even for simple graphs with a universal vertex whose deletion leaves a connected bipartite graph. This answers the variable-threshold complexity question posed by Sebő and by Benchetrit–Sebő. The reduction is from explicitly encoded Set Cover. It uses the classical factor-critical formulation of β , a cone identity, and a block-loss inequality that permits arbitrary partial blocks. In particular, it does not assume an optimal subgraph is a union of whole blocks.

1 Question and conventions

All graphs are finite and undirected. The general input graph is loopless, may have parallel edges, and is 2-vertex-connected: it has at least three vertices and remains connected after deleting any one vertex. A circuit is a connected 2-regular subgraph. An *open ear decomposition* consists of an initial circuit followed by paths with distinct old endpoints and new internal vertices. Each edge occurs in exactly one ear, and the ears partition the edge set. Paths of one edge are allowed. An ear is odd if its number of edges is odd. The initial circuit is also counted as an ear.

For such a graph, $\beta(G)$ is the maximum length of the initial consecutive odd prefix. Once the first even ear occurs, later odd ears do not count. If no odd initial circuit is possible, $\beta(G) = 0$. Sebő’s Question 4 in the 2014 Oberwolfach report [1, printed p. 2914] asks for the complexity of computing this parameter. Benchetrit–Sebő give these precise open-ear conventions, the factor-critical interpretation, and the polynomial-computability question [2, §§1.1,3.1, Question 3.3].

Theorem 1. *The language of pairs (G, t) with G as above, t an integer in binary, and $\beta(G) \geq t$, is NP-complete. NP-hardness holds on simple 2-vertex-connected graphs with a specified universal vertex z such that $G - z$ is connected and bipartite.*

Thus an exact polynomial-time algorithm for β would imply $P = NP$. The threshold varies with the input. This does not assert hardness at any fixed threshold, and is consistent with the fixed-parameter result in [2]. It concerns neither closed ears nor the total number of odd ears throughout a decomposition.

2 The cone identity

A graph H is *factor-critical* if $H - v$ has a perfect matching for every $v \in V(H)$. We use the classical theorem of Lovász: a 2-vertex-connected graph is factor-critical if and only if it has an all-odd open ear decomposition [3, 2]. This is an established structural input, not a new result of this note.

Lemma 2. *For a 2-vertex-connected graph G ,*

$$\beta(G) = \max_H (|E(H)| - |V(H)| + 1), \quad (1)$$

where H ranges over 2-vertex-connected factor-critical subgraphs of G , with value zero if the family is empty. The maximum can be taken over induced subgraphs.

Proof. The union of an all-odd prefix is 2-vertex-connected and factor-critical by the classical theorem. Each ear contributes one to $|E| - |V| + 1$, including the initial circuit. This proves one inequality.

Conversely, an eligible H has an all-odd open decomposition by the same theorem. It extends to G : if a component of $G - V(H)$ had only one distinct neighbor in H , that neighbor would be a cut vertex of G . Hence any such component has two distinct old neighbors; a path through the component between them supplies an open ear. Repeat until all vertices are present, then add all unused edges as one-edge ears. This also proves extension starting from any later intermediate union. The resulting decomposition of G begins with all the ears of H .

For a fixed vertex set, adding its missing edges preserves both factor-criticality and 2-vertex-connectivity and can only increase the quantity in (1). Thus induced H suffice. $\square$

For a graph B , its *cone* $z * B$ adds one new vertex z and one edge from z to every vertex of B .

Lemma 3. *If B is connected and bipartite with at least two vertices, then $G = z * B$ is 2-vertex-connected, and*

$$\beta(G) = \max\{|E(B[S])| : S \neq \emptyset, B[S] \text{ is connected and has a perfect matching}\}. \quad (2)$$

Proof. Deleting z leaves connected B ; deleting any other vertex leaves connectivity through z . A 2-vertex-connected factor-critical graph cannot be bipartite. Indeed its order is odd, its bipartition classes are nonempty, and deleting a vertex from the smaller class leaves unequal class sizes, which precludes a perfect matching. Thus every eligible H in (1) contains z . Make it induced and write $H = z * B[S]$.

Deleting z shows that 2-vertex-connectivity of H requires $B[S]$ connected, and factor-criticality requires a perfect matching of $B[S]$. Conversely, if $B[S]$ is connected with a perfect matching M , then $|S| \geq 2$, and the cone is 2-vertex-connected. Deleting z uses M . After deleting $u \in S$, remove its matching edge uv and match v to z instead. This proves factor-criticality. Finally,

$$|E(H)| - |V(H)| + 1 = (|E(B[S])| + |S|) - (|S| + 1) + 1 = |E(B[S])|.$$

Both directions give (2). Since B contains an edge, the feasible family is nonempty. $\square$

3 The reduction

We reduce from Set Cover, whose NP-completeness is classical [4]. An instance explicitly lists a universe $U = \{1, \dots, m\}$, sets $A_1, \dots, A_n$, and a budget s . It asks whether at most s sets cover U .

Preprocessing. Budgets below zero give a NO instance. With a nonnegative budget an empty universe gives YES. For a nonempty universe, no sets, an uncovered element, or budget zero give NO. If the union is U and $s \geq n$, the answer is YES. These tests are polynomial. Map any trivial YES to a triangle with threshold 1 and any trivial NO to a triangle with threshold 2. A triangle has $\beta = 1$ and is the cone over K_2 , so *every* output remains in the restricted class of Theorem 1. It now suffices to handle

$$m, n \geq 1, \quad 1 \leq s \leq n, \quad \bigcup_{j=1}^n A_j = U.$$

Empty or repeated sets are allowed in this nontrivial input.

Put

$$T = \sum_{j=1}^n (1 + |A_j|), \quad k = T + n + 3, \quad M = (m + 1)k^2, \quad F = M - ks. \quad (3)$$

Here k is a block size; the output threshold is F . We shall use

$$k > T, \quad k > s, \quad k^2 > ks + T, \quad F > 0. \quad (4)$$

Indeed $k - s \geq T + 3$, so $k(k - s) > T$.

Construction. Take pairwise disjoint copies $Q_0, \dots, Q_m$ of $K_{k,k}$, with left and right classes L_i, R_i . Choose a distinguished right vertex $q_i \in R_i$. For each A_j , add a new left vertex c_j , adjacent to q_0 and to each q_i with $i \in A_j$. There are no other interblock edges. Call this graph B . It is simple and bipartite, with

$$|V(B)| = 2k(m + 1) + n, \quad |E(B)| = M + T.$$

Every connector c_j meets the root block Q_0 ; every element block has a connector neighbor since the sets cover U . Thus B is connected. Output

$$(G, t) = (z * B, F). \quad (5)$$

It belongs to the claimed simple cone class.

Proposition 4. *The Set Cover instance is YES if and only if $\beta(G) \geq F$.*

Proof. Suppose first that a cover of size at most s exists. Delete redundant sets to obtain an inclusion-minimal cover indexed by J , and put $r = |J| \leq s$. Each $j \in J$ has a *private element* $p(j)$: an element of A_j belonging to no other selected set. Otherwise A_j could be deleted. These private elements are distinct.

Select all block vertices and the r connectors indexed by J , but omit one left vertex from $L_{p(j)}$ for each $j \in J$. Call the resulting vertex set S . Match c_j to $q_{p(j)}$. In each such block, the remaining unmatched vertices form $K_{k-1, k-1}$; match them internally. Match all other blocks internally as $K_{k,k}$. Distinct private elements make these matches disjoint, and they cover every selected vertex.

Each retained block is connected since $k \geq 2$. Every selected connector meets the root block, and coverage links every element block to one of those connectors. Thus $B[S]$ is connected. Exactly r left block vertices have been omitted, in distinct blocks, losing exactly kr internal edges. All right vertices remain, so every incidence of a chosen connector remains. Consequently

$$|E(B[S])| = M - kr + \sum_{j \in J} (1 + |A_j|) \geq M - ks = F.$$

Lemma 3 gives $\beta(G) \geq F$.

Conversely, use that lemma to choose S for which $B[S]$ is connected, has a perfect matching, and has at least F edges. Let r be its number of selected connectors. For every block, set

$$a_i = |L_i \setminus S|, \quad b_i = |R_i \setminus S|, \quad d_i = k(a_i + b_i) - a_i b_i.$$

Here $0 \leq a_i, b_i \leq k$, and d_i is exactly the number of lost internal edges of Q_i . No whole-block assumption has been made.

Every block retains at least one internal edge. Otherwise at most $M - k^2$ internal edges remain and the total connector contribution is at most T , giving

$$|E(B[S])| \leq M - k^2 + T < M - ks = F$$

by (4), a contradiction. In particular, both the root block and every element block have selected internal edges.

The perfect matching forces the selected bipartition classes to have equal size. All connectors are left vertices, so

$$\sum_i (k - a_i) + r = \sum_i (k - b_i), \quad \text{hence} \quad \sum_i (a_i - b_i) = r.$$

The exact identity

$$d_i - k(a_i - b_i) = b_i(2k - a_i) \geq 0 \tag{6}$$

therefore gives $\sum_i d_i \geq kr$. There are only T connector edges in the whole graph. Thus

$$M - ks \leq |E(B[S])| \leq M - kr + T, \quad k(r - s) \leq T < k.$$

Integrality implies $r \leq s$.

Finally, fix an element i . Its block has a selected internal edge, and so does the distinct root block. Since $B[S]$ is connected, a path from the element block to the root must leave Q_i through a selected connector c_j . Such an edge exists only when $i \in A_j$. The r selected sets therefore cover every element, within budget. $\square$

Encoding size. Let L be the length of the explicit Set Cover input. After the union test, $m, n, T = O(L)$, and hence $k = O(L)$. The construction lists $O(L^2)$ vertices and $O(L^3)$ edges; the binary threshold F has polynomial bit length. Renumbering the elements is polynomial. No succinct multiplicities, weights, implicit graph, or parallel edges are used. Proposition 4 proves NP-hardness.

Membership in NP. For $t > 0$, a certificate supplies a subgraph $H \subseteq G$ and a perfect matching of $H - v$ for each $v \in V(H)$. Check that $|V(H)| \geq 3$, that H and every $H - v$ are connected, that all matching edges belong to the required deletion graph and cover every remaining vertex exactly once, and that $|E(H)| - |V(H)| + 1 \geq t$. These are polynomial checks with a polynomial certificate. Lemma 2 proves the required equivalence. For $t \leq 0$ the answer is immediate on valid inputs. Invalid input graphs may first be rejected in polynomial time. This completes the proof of Theorem 1.

4 Relation to earlier work and verification

The cone identity optimizes the number of *all induced edges* on a *connected* perfectly matchable vertex set. These two qualifications matter. Sharifov’s polynomial algorithm [5] permits disconnected perfectly matchable selections. For example, take three $K_{3,3}$ blocks and two left connectors, one joining the distinguished root right vertex to each of the other blocks. All block vertices alone give 27 induced edges and a perfect matching, but three components. The connected optimum is 25: retaining all three blocks requires both connectors, whose matching balance loses at least six block edges and adds at most four connector edges. The bound is attained by omitting one left vertex in each nonroot block and matching the connectors there. If a block retains no internal edge, at most $27 - 9 + 4 = 22$ edges remain; with fewer than two connectors one cannot connect all three retained blocks. Thus connectivity cannot simply be dropped.

Weighted Connected Matching [6] instead maximizes the weight of the *matching edges*; it does not maximize the number of all edges induced by the endpoints. Its related hardness theorem is therefore credited without identifying the two objectives. Constrained ear decompositions [7] and joins in matroids [8] likewise concern different constraints or total-parity parameters. The new ingredient here is the explicit Set Cover reduction for (2), including the partial-block inequality (6).

The original question was stated as open in 2014 and 2015 [1, 2]. A bounded primary-source and publication-register investigation through 22 October 2026 authenticated no earlier publication of the full theorem above. This is not an exhaustive priority or exclusive-firstness certification, and the question’s source and the cited dependencies are not represented as publications of this reduction. Exact source versions and scope limits are recorded in the accompanying `DEPENDENCIES.md` and `PRIORITY.md`.

The accompanying standard-library Python diagnostics test the original complete-ear prefix objective on finite graphs against independent matching and induced-subgraph enumerations, the exact omission identity, explicit full-size reduction witnesses, preprocessing cases, and falsifiable controls. These finite tests supplement the proof; they do not establish its universal quantifiers or certify priority. Reproduction instructions and exact output are included in the source archive.

AI involvement and status. AI tools were extensively involved in mathematical research, proof development and reconstruction, computational verification, literature investigation, manuscript preparation, editing, and adversarial auditing. Alec Kriebel is the named human author. This manuscript is a preprint and has not undergone formal independent human peer review. Internal AI reviews are not such peer review or proof-assistant verification. Authored prose is licensed CC BY 4.0 and authored code is MIT licensed.

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