A characteristic obstruction in higher Koszul homology

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Abstract

Over every field of positive characteristic, the free algebra on two degree-one generators is Koszul but has nonzero positive-degree higher Koszul homology. We give an explicit class and the complete cyclic-word calculation. This elementary application of classical tensor-algebra formulas refutes the unrestricted-field equivalence reported in Oberwolfach Report 10/2016. The characteristic-zero converse remains unresolved here.

1 The unrestricted statement and its scope

Solotar's contribution to Oberwolfach Report 10/2016 [1, printed pp. 479–481] asks whether a quadratic algebra is Koszul if and only if its higher Koszul homology vanishes in positive degrees. The contribution states no restriction on the characteristic. We use higher homology with coefficients in the algebra itself and fundamental cocycle e_A, as defined by Berger–Lambre–Solotar [2, §5.2]. Read over unrestricted fields, the equivalence is false.

This qualification matters. The forward theorem in [2, Theorem 6.4] assumes characteristic zero; the adjacent Conjecture 6.5 includes the additional condition $\mathrm{HK}_0^{\mathrm{hi}}(A) \cong k$ and is motivated as a converse, while its preceding discussion also considers removing the characteristic hypothesis. Our counterexample refutes the unrestricted forward implication of either formulation. It neither proves nor disproves the characteristic-zero vanishing-to-Koszulness converse, and does not establish a unique historical intended domain.

The calculation is an elementary assembled consequence of the classical tensor-algebra Hochschild formulas of Loday–Quillen [3, §5], Goodwillie's Euler cap operator [4, II.4.8], and the BLS definition. No new homological mechanism or historical firstness is claimed. We record the explicit obstruction to the unrestricted written statement.

2 The free-algebra calculation

Let k be a field, let V have a finite basis of r letters, and put $A = T_k(V)$, graded by word length. This is a quadratic algebra with relation space $R = 0$. Write $\mathrm{HK}_i^{\mathrm{hi}}(A)[n]$ for total internal weight n: a chain in $A_{n-i} \otimes W_i$ has that total weight. For $n > 0$, let $\mathcal{O}_n$ denote the cyclic orbits of length-n words in the chosen letters. Orbit cardinality means the number of *distinct* rotations, which can be smaller than n.

Proposition 1 (Classical specialization)**.** *The algebra A is Koszul over every field. If* $\operatorname{char} k = p > 0$*, then, for $n > 0$ and $i = 0, 1$,*

$$\mathrm{HK}_i^{\mathrm{hi}}(A)[n] \cong \bigoplus_{\substack{O \in \mathcal{O}_n \\ p \mid\mid |O|}} k.$$

In characteristic zero these positive-weight groups vanish. For all fields, $\mathrm{HK}_i^{\mathrm{hi}}(A) = 0$ for $i \geq 2$, $\mathrm{HK}_0^{\mathrm{hi}}(A)[0] = k$, and $\mathrm{HK}_1^{\mathrm{hi}}(A)[0] = 0$.

*OpenAI tools were used extensively in development, solving, drafting, coding and adversarial verification. This is an unrefereed preprint; no completed external human peer review or formal proof-assistant verification is claimed.

Proof. Multiplication $\mu(a \otimes v) = av$ maps the word/letter basis of $A \otimes V$ bijectively onto the nonempty words of A. Consequently

$$0 \longrightarrow A \otimes V \xrightarrow{\mu} A \xrightarrow{\varepsilon} k \longrightarrow 0$$

is an exact linear graded free left resolution, proving Koszulness. Here ε is augmentation. This map μ is distinct from the coefficient-A commutator differential used next.

The quadratic Koszul spaces satisfy $W_0 = k$, $W_1 = V$, and $W_j = 0$ for $j \geq 2$, since every defining intersection contains $R = 0$. Thus the ordinary coefficient complex is exactly

$$0 \longrightarrow A \otimes V \xrightarrow{b} A \longrightarrow 0, \qquad b(a \otimes v) = av - va.$$

In particular $\mathrm{HK}_1(A) = \ker b$, $\mathrm{HK}_0(A) = \operatorname{coker} b$, and $\mathrm{HK}_j(A) = 0$ for $j \geq 2$. The fundamental left cap product on $a \otimes v$ is va. For a cycle $z = \sum a_j \otimes v_j$, the equation $b(z) = 0$ gives $\sum v_j a_j = \sum a_j v_j$. Hence the induced higher differential is

$$\delta : \ker b \longrightarrow \operatorname{coker} b, \qquad \delta[z] = \Big[\sum_j a_j v_j\Big].$$

This definition is valid also in characteristic two [2, §5.2]. There is no incoming higher boundary from $\mathrm{HK}_2(A)$.

Fix $n > 0$. Concatenation identifies $A_{n-1} \otimes V$ and A_n separately with the vector space E_n on length-n words. Let ρ move the final letter to the beginning. Then $b = 1 - \rho$; this is the classical small complex [3, Lemmas 5.1–5.2]. The higher map becomes the natural map

$$E_n^{\rho} \longrightarrow (E_n)_{\rho} := E_n/(1-\rho)E_n.$$

On an orbit O of cardinality d, equality of neighboring coefficients makes the invariant subspace the line generated by $s_O = \sum_{w \in O} w$. The coinvariant quotient is the line generated by any $[w]$, which is nonzero: the coefficient-sum functional kills all differences and takes $[w]$ to one. No division by d is involved. The induced map is

$$s_O \longmapsto d[w].$$

Its kernel and cokernel are both one-dimensional precisely when $d = 0$ in k, and otherwise both vanish. Summing the orbit blocks proves the formula. At weight zero only the degree-zero chain k occurs; all assertions in degrees at least two follow from the absence of their ordinary homology groups. □

Corollary 2. *For every field k of positive characteristic p, the quadratic algebra $k\langle x, y\rangle$ is Koszul and* $\mathrm{HK}_1^{\mathrm{hi}}(k\langle x, y\rangle)[p] \neq 0$.

Proof. The word $w = x^{p-1}y$ has exactly p distinct rotations $w_j = \rho^j(w)$, because the unique y changes position. Write $w_j = a_j v_j$, with v_j its final letter. The chain

$$z_p = \sum_{j=0}^{p-1} a_j \otimes v_j$$

is a sum of distinct basis tensors and is therefore nonzero. Its commutator boundary telescopes, and $\delta[z_p] = p[w] = 0$. There are neither ordinary boundaries entering degree one nor higher boundaries entering from HK_2. Thus $[z_p] \neq 0$ in higher degree one. In characteristic two, $z_2 = x \otimes y + y \otimes x$. □

The same orbit contributes a positive-weight higher degree-zero class, so adding $\mathrm{HK}_0^{\mathrm{hi}}(A) \cong k$ cannot rescue the unrestricted equivalence. In contrast, one generator has only singleton orbits; its positive-weight higher groups vanish in every characteristic. In particular, listing repeated rotations of x^p cannot supply a counterexample.

3 Classical operators, verification and priority limits

The operator here is Euler cap, not Connes' differential B. Loday–Quillen [3, Lemma 5.3] identify B with the cyclic norm in the opposite direction. On an orbit of size d in weight n, Euler cap has scalar d, whereas B has scalar n/d. Their composites multiply by n, consistently with Goodwillie's identity [4, Theorem II.4.2]. For the primitive weight-p orbit above, B is invertible in characteristic p while Euler cap is zero. The two homologies cannot be substituted for each other merely from characteristic dependence.

The supporting archive includes editable source, all 65 frozen original files and four evidence archives, independent algebra and source-scope reviews, and a dated bounded priority audit. Fresh exact controls compare 234 matrix/necklace cases, check 14 prime witnesses and ten negative controls, and run 2110 explicit checks in both normal and optimized Python. Corrupted expectations are rejected in both modes. These finite and integrity checks supplement the unlimited algebraic proof; they do not certify it or determine historical priority.

No earlier explicitly stated exact counterexample was located in the inspected corpus. This bounded finding does not establish worldwide firstness or current openness. The entire mechanism receives the classical credit stated above. Version and access limits, including unavailable complete journal-final bodies for some related work, are recorded in the accompanying audit. Downloaded primary publications are excluded from the deposit; bibliographic locators and hashes remain.

References

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