An interior second-order defect from a Hele–Shaw contact arc

Alec Kriebel
Independent researcher
ORCID: 0009-0001-9320-500X

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Abstract

We give a detailed application of Ross–Witt Nyström's smooth Hele–Shaw contact-arc construction: the associated homogeneous complex Monge–Ampère solution on $\mathbb{P}^1 \times \mathbb{D}$ has no second-order real Taylor expansion at an interior point, despite smooth strictly Kähler fibre boundary data. Filling a zero-area contact arc gives a strict Green-function time-slope gap; smooth variation on the covering surface supplies finite nonzero curvature on the other side. Legendre duality then yields incompatible quadratic coefficients. The 2021 Oberwolfach report already announces an interior construction. This is an attributed proof/application note; historical priority of the deduction is unresolved and no first-discovery claim is made.

1 Statement and attribution

Let $\mathbb{D} = \{\tau \in \mathbb{C} : |\tau| < 1\}$ and let $\pi : \mathbb{P}^1 \times \mathbb{D} \to \mathbb{P}^1$ be projection. We use the coherent normalization

$$d^c = \frac{i}{4\pi}(\bar\partial - \partial), \qquad dd^c = \frac{i}{2\pi}\partial\bar\partial, \qquad dd^c \log|z|^2 = \delta_0, \qquad \omega_{\mathrm{FS}} = dd^c \log(1+|z|^2), \quad \int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1.$$

A second-order real Taylor expansion at p means an expansion in real local coordinates of the form $u(p+h) = u(p) + \ell(h) + Q(h) + o(|h|^2)$, where ℓ is linear and Q is quadratic.

Theorem 1. *There exists $\phi \in C^\infty(\mathbb{P}^1)$ with $\omega_\phi := \omega_{\mathrm{FS}} + dd^c\phi > 0$ such that the bounded weak solution Φ of*

$$\pi^*\omega_{\mathrm{FS}} + dd^c\Phi \geq 0, \qquad (\pi^*\omega_{\mathrm{FS}} + dd^c\Phi)^2 = 0, \qquad \Phi(z,\tau) = \phi(\tau z) \quad (|\tau| = 1) \tag{1}$$

has no second-order real Taylor expansion at a point (z_*, τ_*) with $0 < |\tau_*| < 1$. The equation is in the Bedford–Taylor sense, and each boundary fibre form is strictly positive. In particular Φ is not C^2 in the interior.

The construction of ϕ, the obstacle-current identity, the smooth strong flow on a covering surface, and the Monge–Ampère/Hele–Shaw duality are all due to Ross–Witt Nyström [1, 2]. The additional argument written here is the arc-filled Green comparison, its combination with finite left curvature, and the resulting interior Taylor defect. Green extremality, the Hopf lemma and convex conjugacy used in that argument are classical tools.

This attribution is material to the status of the result. Their survey [5, §4.3] discusses the interior-location question in its dated 2017/2018 context. Witt Nyström's joint-work report with Ross [6, pp. 1323–1324] already announces and sketches an interior construction, using a terminal-domain harmonic-disc route and an inverse-flow density whose smoothness is left tentative there;

that discussion permits vanishing density at the terminal singular points. It must receive credit for the announced phenomenon. The present note uses instead a finite-time contact arc and the strictly positive form supplied by [1, Proposition 3.4]. The earlier published contact theorem [1, Theorem 6.2] states its nonsmoothness conclusion on the boundary circle. The rank-degeneration examples in [4] by themselves do not give the Taylor defect below.

A bounded literature audit through 15 October 2026 found no complete exact predecessor in the recorded inspected scope. It does not establish historical firstness, independent discovery, or present worldwide openness. In particular, the Oberwolfach contribution is not being recast as a numbered unresolved conjecture. This note supplies a checkable proof application, rather than a claim to a newly discovered general phenomenon. Source versions, precise inspection scope and remaining coverage/access gaps are in the supplement.

2 Credited flow and duality inputs

For a potential ϕ as above, set

$$\psi_t = \sup\{\psi \in \mathrm{PSH}(\mathbb{P}^1, \omega_{\mathrm{FS}}) : \psi \le \phi,\ \nu_0(\psi) \ge t\}, \qquad \Omega_t = \{\psi_t < \phi\}, \quad 0 \le t \le 1.$$

Here ν_0 is the coefficient of the unit logarithmic pole, with the usual upper-semicontinuous envelope convention. The domains increase with t. The envelopes decrease and are concave in t. For $0 < t < 1$,

$$\omega_{\mathrm{FS}} + dd^c\psi_t = \mathbf{1}_{\Omega_t^c}\omega_\phi + t\delta_0, \qquad \int_{\Omega_t} \omega_\phi = t, \tag{2}$$

and ψ_t is $C^{1,1}$ away from the pole [1, Proposition 2.5].

We use the following specific features of the published construction, not an assertion about arbitrary self-contacting flows.

Proposition 2 (Ross–Witt Nyström construction)**.** *One can choose a smooth strictly positive ω_ϕ and $T \in (0,1)$ so that $\Omega = \Omega_T$ is connected and its boundary has a smooth contact segment S of positive length. At a relative interior point $q \in S$, a sufficiently small coordinate disc B satisfies $B \setminus S \subset \Omega$, with fluid on both sides of S. The pre-contact domains are projections of a smooth family of smoothly bounded simply connected strong-flow domains Ω'_t on a fixed covering surface, extending smoothly through $t = T$. Projection is conformal and injective on the open domains for $t \le T$ close to T. The limiting outward boundary velocity on the lift is strictly positive.*

These are the construction in [1, Proposition 3.4 and its proof], with the strong-flow extension from its Theorem 2.15 (and the two-sided local flow discussed in [3, Remark 7.1]). Nonvanishing limiting velocity and the two-sided fluid geometry are recorded in the proof of [1, Theorem 6.2]. The covering resolves the contact: distinct boundary points can project to the same arc. Injectivity is required only on the open fluid domain, not on its closure. A smooth compact domain and collar on the cover can be placed in a conformal coordinate chart; smooth Green dependence is then the uniformly elliptic Dirichlet dependence of [1, Appendix A].

Write $G_U(w,0)$ for the negative Green function with singular part $\log|w|^2$ at the injection point. The duality [1, Theorem 4.4] gives

$$\widetilde{\Phi}(w,\tau) = \sup_{0\le t\le 1} \{\psi_t(w) + (1-t)\log|\tau|^2\}. \tag{3}$$

For $\tau \neq 0$, [1, Proposition 4.2] identifies it with the bounded solution in (1) through

$$\Phi(z,\tau) + \log|\tau|^2 + \log(1+|z|^2) = \widetilde{\Phi}(\tau z,\tau) + \log(1+|\tau z|^2). \tag{4}$$

We use this identity in finite affine coordinates near the point in question. The solution is the usual envelope with the prescribed smooth boundary data. A constant $\min \phi$ is an admissible lower bound; vertical subharmonic comparison gives the upper bound $\max \phi$. The cited duality concerns this weak solution, with its boundary trace, not just a formal transform.

3 A strict Green-function slope gap

Choose q, B as in Proposition 2, away from 0 and the arc endpoints, and put $D_+ = \Omega \cup B$. This is an open connected proper domain. Its difference from Ω has area zero, whereas its complement has positive ω_ϕ-area $1 - T$. Thus both domains are Greenian.

Lemma 3. *For every $w \in \Omega \setminus \{0\}$,*

$$\partial_t^+ \psi_T(w) \le G_{D_+}(w, 0) < G_\Omega(w, 0).$$

Proof. On $D_+ \setminus \{0\}$, (2) makes ψ_T plus a local potential of ω_{FS} distributionally harmonic: the newly filled part of S has area zero, and the current identity supplies the entire measure, including the absence of a line measure. Weyl regularity therefore gives harmonic continuation across the arc. This is not an arbitrary removability assertion for positive-capacity sets.

For $0 < h < 1 - T$ let $v_h = (\psi_{T+h} - \psi_T)/h$. Then $v_h \le 0$ and, on D_+,

$$dd^c v_h = \delta_0 + h^{-1} \mathbf{1}_{\Omega^c_{T+h}} \omega_\phi \ge \delta_0.$$

In fact monotonicity of the domains makes the extra term zero almost everywhere on D_+; the weaker inequality suffices. The logarithmic coefficient at 0 is one. The extremal characterization of the negative Green function gives $v_h \le G_{D_+}$.

Domain monotonicity gives $G_{D_+} \le G_\Omega$. The inequality is strict: the old Green function tends to zero at a smooth interior point of the contact slit from either side, as is also seen on the smooth lift. Meanwhile $G_{D_+}(q, 0) < 0$ since q is now an interior point. Their nonnegative harmonic difference on Ω, with the equal logarithmic poles cancelled, is nonzero. The strong maximum principle makes it positive everywhere. Finally $t \mapsto \psi_t(w)$ is finite and concave near T, so the right derivative exists and passing $h \downarrow 0$ proves the claim. □

The arc must be nonpolar. Filling a single isolated point would not justify this strict comparison. No description, smoothness or filling-rate estimate for the post-contact weak flow was used.

4 Finite nonzero curvature from the smooth lift

Fix a finite affine point $w_* \in \Omega \setminus \{0\}$ and write $a(t) = \psi_t(w_*)$. It belongs to the smoothly evolving domains for all $t < T$ sufficiently close to T.

Lemma 4. *The pre-contact branch satisfies*

$$a'_-(T) = \kappa_0 := G_\Omega(w_*, 0) < 0, \qquad a''_-(T) = L \in (-\infty, 0),$$

with smooth one-sided dependence up to T.

Proof. On a smooth pre-contact domain, differentiating the harmonic Dirichlet problem for ψ_t gives

$$\partial_t \psi_t(w) = G_{\Omega_t}(w, 0).$$

Indeed this derivative is harmonic away from its fixed unit logarithmic pole. Its boundary value is zero: differentiating $\psi_t - \phi = 0$ along the moving boundary has no spatial contribution, by the C^1 smooth fit. Uniqueness of the Green Dirichlet problem proves the identity. The smooth Dirichlet problem on the lifted family justifies these differentiations up to the one-sided limit at T.

Choose fixed lifts of 0 and w_* under the fixed covering projection. For $t \le T$ the lifted and original Green functions agree by conformal invariance. Smooth domain dependence on the lift gives a finite derivative $H = \partial_t G_{\Omega'_t}|_T$, harmonic even at the pole after removal of its time-independent logarithmic singularity. Differentiating its zero boundary value gives

$$H = -V \partial_n G_{\Omega'_T} \quad \text{on } \partial\Omega'_T.$$

Here $V > 0$ is the limiting outward normal velocity. The outward normal derivative of the negative Green function is positive by Hopf's lemma. Thus $H < 0$ on the boundary and throughout the domain, and evaluation at the fixed lift of w_* gives finite $L < 0$. The projected singular boundary is never used as a smooth Dirichlet boundary at T. □

5 Legendre plateau and incompatible Taylor coefficients

Lemma 3 supplies

$$a'_+(T) \le b := G_{D_+}(w_*, 0) < \kappa_0 = a'_-(T). \tag{5}$$

For κ near κ_0 set

$$F(\kappa) = \widetilde{\Phi}(w_*, e^{\kappa/2}) = \kappa + \sup_{0 \le t \le 1} \{a(t) - \kappa t\}.$$

Global concavity is essential. For $b < \kappa < \kappa_0$, the supporting slopes in (5) make T the unique global maximizer, so

$$F(\kappa) = a(T) + (1 - T)\kappa. \tag{6}$$

For $\kappa > \kappa_0$ sufficiently close, Lemma 4 and the implicit function theorem give a unique $t(\kappa) < T$ near T with $a'(t(\kappa)) = \kappa$. It too is the unique global maximizer: a supporting line for a concave function is global, and local strict curvature excludes any second point of equality. Thus no unknown post-contact branch can overtake it. Endpoint values in the supremum cause no difficulty, even if interpreted in the extended-real envelope convention.

Put $\delta = \kappa - \kappa_0$ and $x = t(\kappa) - T$. As $\delta \downarrow 0$,

$$x = \delta/L + o(\delta), \qquad a(T + x) = a(T) + \kappa_0 x + \tfrac{1}{2} L x^2 + o(x^2).$$

Substitution, together with (6) on the other side, yields

$$F(\kappa_0 + \delta) - F(\kappa_0) - (1 - T)\delta = \begin{cases} 0, & \delta < 0 \text{ sufficiently small}, \\ -\delta^2/(2L) + o(\delta^2), & \delta > 0. \end{cases} \tag{7}$$

Since $L < 0$, no common quadratic Taylor coefficient exists. This proves failure of a second-order expansion, rather than merely a discontinuous limiting Hessian.

Completion of Theorem 1. The number κ_0 is finite and negative. Define $\tau_* = e^{\kappa_0/2} \in (0, 1)$ and $z_* = w_*/\tau_*$. Along the smooth curve $(z, \tau) = (w_* e^{-\kappa/2}, e^{\kappa/2})$, (4) gives

$$\Phi(w_* e^{-\kappa/2}, e^{\kappa/2}) = F(\kappa) + \log \frac{1 + |w_*|^2}{e^\kappa + |w_*|^2}.$$

The added term is smooth. A second-order real Taylor expansion for Φ at (z_*, τ_*) would restrict to one along this curve, contradicting (7). Finally $(z, \tau) \mapsto \phi(\tau z)$ is jointly smooth on $\mathbb{P}^1 \times \partial\mathbb{D}$, and circle rotations preserve ω_{FS}, so each boundary fibre form is strictly positive. □

Scope and verification. This is an existence recipe using a credited designer-flow theorem, not a closed-form potential. The argument requires a positive-length arc, $0 < T < 1$, a non-pole interior observation point, the smooth lifted family, and finite strictly negative L. It does not prove an arbitrary point tangency or terminal-time slit has this defect. The supplementary exact rational verifier checks the scalar support and quadratic signs, including invalid-hypothesis controls; it does not certify the geometric PDE inputs by computation. The proof above supplies those analytic deductions from the cited results.

AI use and review status. AI tools were used extensively in investigating the problem, deriving, drafting and checking the argument, auditing sources and preparing the verification materials. Independent AI adversarial reviews are recorded with their actual scope in the repository. This preprint is unrefereed and has not undergone conventional human peer review. The author is responsible for the claims and their qualifications.

References

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