A first-order obstruction to Habiro's rank interpolation

Alec Kriebel
Independent researcher
ORCID: 0009-0001-9320-500X

16 October 2026 Version 1.0

Abstract

The rank completion printed in Ohtsuki's Conjecture 7.30 cannot interpolate the standard unified $\mathfrak{sl}_n$ invariants of an integral homology sphere with nonzero Casson invariant. A finite quotient forces its first Taylor coefficients to be affine in n, contrary to the classical cubic coefficient formula. The specified Poincaré sphere gives the incompatible values $0, -6, -24$. This is a negative answer to the literal displayed statement, obtained from established quantum-invariant results.

1 The printed completion

In the notation of Ohtsuki's collection [1, pp. 490–492], put

$$R = \varprojlim_m \mathbb{Z}[q, q^{-1}]/\Big(\prod_{j=1}^{m}(q^j - 1)\Big), \qquad E_m(t) = \prod_{j=1}^{m}(t - q^j), \qquad S = \varprojlim_m R[t]/(E_m(t)).$$

The transition maps are quotient maps. This R is the ordinary integral Habiro ring: q is already a unit in every polynomial quotient, since the defining polynomial has constant term ± 1. For $F \in S$, evaluation at $t = q^n$ is defined through any coordinate $m \geq n$; compatibility makes the value independent of that choice.

Conjecture 7.30 asks for an invariant $F_M \in S$ of every integral homology 3-sphere M such that

$$F_M(q^n) = I_n(M) \quad (n \geq 1), \qquad I_n(M) = I^{\mathfrak{sl}_n}(M) \ (n \geq 2), \quad I_1(M) = 1.$$

Thus n denotes matrix size, not the Lie rank $n - 1$. Its fixed-Lie-algebra premise, Conjecture 7.29, is supplied by the Habiro–Lê unification theorem [3, Theorem 1.1].

Theorem 1. *Use the coefficient ring and standard invariant normalization above. If $\lambda_{\mathrm{C}}(M) \neq 0$, there is no $F_M \in S$ with these evaluations. Here λ_{C} is the Casson–Walker invariant in Lê's Lescop normalization. In particular, the orientation of the Poincaré homology sphere given by -1 surgery on the left-handed trefoil is a counterexample to the printed conjecture.*

2 A finite-jet constraint

Let $D = \mathbb{Z}[x]/(x^2)$. The substitution $q \mapsto 1 + x$, $q^{-1} \mapsto 1 - x$ kills $(q - 1)(q^2 - 1)$. Projection to the second coordinate therefore defines a ring map $T_1 : R \to D$. No injectivity assertion for Taylor expansion is needed.

Lemma 1. *For $F \in S$, write $T_1(F(q^n)) = u_n + v_n x$ for $n = 1, 2, 3$. Then $u_1 = u_2 = u_3$ and $v_3 - 2v_2 + v_1 = 0$. Consequently, $F(q) = 1$ implies $v_3 = 2v_2$.*

Proof. The third coordinate of F has a representative $P(t) = a + bt + ct^2$, with $a, b, c \in R$, by division by the monic polynomial E_3. Evaluation on this quotient is well-defined at q, q^2, q^3. Write $T_1(a) = a_0 + a_1 x$, and similarly for b, c. Since $(1 + x)^k = 1 + kx$ in D,

$$T_1(P(q^n)) = a_0 + b_0 + c_0 + (a_1 + b_1 + c_1 + n(b_0 + 2c_0))x.$$

The coefficient of x is affine in n. If $F(q) = 1$, its value at $n = 1$ is zero, which gives the last assertion. □

Only one finite interpolation quotient and one finite coefficient quotient occur in this argument; it differentiates no infinite sum.

3 The quantum coefficient and counterexample

Habiro–Lê identify the Taylor series of the unified invariant with the Ohtsuki series [3, Proposition 1.9]; their Proposition 1.2 also gives the projective specialization. Lê's published first-coefficient formula [2, p. 821], in the variable $x = q - 1$, specializes on an integral homology sphere to

$$[x]\, T_1(I_n(M)) = n(n^2 - 1)\lambda_{\mathrm{C}}(M), \qquad n \geq 2. \tag{1}$$

The factor involving $|H_1(M; \mathbb{Z})|$ is 1 here. These sources use a common type-A parameter: roots have squared length 2, and $q = v^2 = \exp(h)$ [3, §3A1–2]. Changing from h to $q - 1$ does not change a first coefficient. Thus (1) concerns the invariants in the printed question, with no rank-dependent rescaling.

If F_M existed, Lemma 1 would give

$$0 = v_3 - 2v_2 + v_1 = 24\lambda_{\mathrm{C}}(M) - 2 \cdot 6\lambda_{\mathrm{C}}(M) = 12\lambda_{\mathrm{C}}(M).$$

This proves the nonexistence assertion in Theorem 1.

For an explicit orientation check, let P be the sphere in that theorem. The source's series [1, p. 492], also given as a unified invariant in [4, §16.1, equation (16.1)], is

$$I_2(P) = \sum_{k \geq 0} q^k \, \frac{\prod_{j=k+1}^{2k+1}(1 - q^j)}{1 - q}.$$

After cancelling $1 - q$ in each polynomial summand, it becomes

$$q^k(1 + q + \cdots + q^k) \prod_{j=k+2}^{2k+1} (1 - q^j).$$

The $k = 0$ term is 1; the $k = 1$ term is $q(1+q)(1-q^3) = -6x + O(x^2)$. Every $k \geq 2$ term is divisible by x^k, so the entire remaining tail contributes no constant or linear term. Hence $[x]I_2(P) = -6$ and (1) fixes $\lambda_{\mathrm{C}}(P) = -1$ in the stated convention. It follows that

$$(v_1, v_2, v_3) = (0, -6, -24), \qquad v_3 - 2v_2 + v_1 = -12 \neq 0.$$

This proves the explicit counterexample. □

4 Scope, attribution, and verification

The obstruction concerns the displayed ordinary completion. It gives no existence criterion when $\lambda_C(M) = 0$. Omitting the assigned value at $n = 1$ does not remove the obstruction: the same finite-quotient argument at sizes $2, 3, 4$ gives a second difference of $18\lambda_C(M)$. Poles in $q - 1$, a different completion, or rank-dependent normalization change the question; no corrected formulation is proposed here.

The cubic type-A behavior is classical, already explicit in Lê–Murakami–Ohtsuki [5, Proposition 5.3 and Remark 5.4] and in the all-rank free-energy formalism [6, §3]. The coefficient and unification inputs above are credited prior results. Fixed-rank unification, link-interpolation stability, and enlarged knot-color completions [7, 8, 9] do not by themselves assert the printed S-valued homology-sphere interpolation. No novelty, firstness, or continued-openness claim is made. The accompanying bounded priority audit records its source-access and coverage gaps. The collection's volume year is 2002; the publisher records its release on 1 June 2004.

Source-free verification materials provide exact arithmetic checks, preserved original files, independent adversarial reports, and reproduction instructions. Portable finite-checker stages were reproduced on macOS; the legacy Linux/bubblewrap whole-filesystem read-only replay was not reproduced there. The cited topology theorems remain mathematical inputs. AI tools were used extensively in developing and checking this work. This note is unrefereed; no independent human peer review or formal proof-assistant certification is asserted.

© 2026 Alec Kriebel. Licensed under CC BY 4.0.

References

[1] T. Ohtsuki (editor), *Problems on invariants of knots and 3-manifolds*, Geom. Topol. Monogr. **4** (2002), 377–572, Conjecture 7.30, p. 492. Publisher record.

[2] T. T. Q. Lê, *On perturbative PSU(n) invariants of rational homology 3-spheres*, Topology **39** (2000), 813–849. doi:10.1016/S0040-9383(99)00037-3.

[3] K. Habiro and T. T. Q. Lê, *Unified quantum invariants for integral homology spheres associated with simple Lie algebras*, Geom. Topol. **20** (2016), 2687–2835. doi:10.2140/gt.2016.20.2687.

[4] K. Habiro, *A unified Witten–Reshetikhin–Turaev invariant for integral homology spheres*, Invent. Math. **171** (2008), 1–81. The equation locator used here refers to arXiv:math/0605314v1 (2006).

[5] T. T. Q. Lê, J. Murakami and T. Ohtsuki, *On a universal perturbative invariant of 3-manifolds*, Topology **37** (1998), 539–574. Author-hosted published text.

[6] S. Garoufalidis, T. T. Q. Lê and M. Mariño, *Analyticity of the free energy of a closed 3-manifold*, SIGMA **4** (2008), 080. Author-hosted text.

[7] A. Beliakova and E. Gorsky, *Cyclotomic expansions for $\mathfrak{gl}_N$ link invariants via interpolation Macdonald polynomials*, Selecta Math. **30** (2024), 101. doi:10.1007/s00029-024-00990-y.

[8] H. Fang and T. Zhou, *Cyclotomic Newton expansions and a rank-uniform integer-valued Newton completion*, arXiv:2608.29585v1 (30 August 2026).

[9] H. Fang, *Quantum color Newton rings*, arXiv:2610.21102v1 (15 October 2026).