

An application of separable-state nonrepresentability to Grassmann orbitopes and calibrations

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Abstract

We apply Fawzi’s established obstruction for normalized complex separable states to the oriented real Grassmann orbitope $C_{4,12}$. Two affine hyperplanes select a section real-linearly isomorphic to $\text{Sep}(3, 3)$, so this mass ball has no finite projected semidefinite lift. A compact first-moment argument then disproves the universal degree-one supporting sum-of-squares assertion formulated by Sanyal, in joint work with Rostalski. It also gives a negative answer in general to Harvey–Lawson’s Question 6.5: some constant comass-one four-form on $\mathbb{R}^{12}$ has no pointwise or local norm completion. The failing unnormalized functional may have rational coefficients, but no explicit coefficient list or exact comass is supplied. The reduction combines classical Kähler faces, established fermionic projector bodies and the prior tensor-to-wedge lemma with Fawzi’s theorem; worldwide priority of this assembly is not certified.

1 The original question and the result

Let $X_{k,n}$ consist of all decomposable unit real k -vectors in $\Lambda^k \mathbb{R}^n$, including both orientations, and let $C_{k,n} = \text{conv}(X_{k,n})$. Write $I(X_{k,n})$ for the full real vanishing ideal. Sanyal’s report [1, pp.708–709] asks whether, for every real linear functional ℓ and $\delta = \max_{X_{k,n}} \ell$, one has

$$\delta - \ell = \sum_j h_j^2 \pmod{I(X_{k,n})}, \quad \deg h_j \leq 1. \quad (1)$$

Thus the h_j may be affine, including constants. That source credits joint computational work with Philipp Rostalski and the equivalence with Harvey–Lawson’s Question 6.5 [2, p.68]. This connection and the first-theta-body formulation already appear in Rostalski’s 2010 slides and Gouveia–Thomas’s chapter [10, 11].

Here a finite semidefinite lift means a linear image of a spectrahedron of any finite size, with arbitrary affine equalities and auxiliary variables. It is stronger to exclude such lifts than to exclude an unprojected LMI or one specified hierarchy. We use the following established theorem: the normalized complex body

$$\text{Sep}(3, 3) = \text{conv}\{xx^* \otimes yy^* : x, y \in \mathbb{C}^3, \|x\| = \|y\| = 1\}$$

has no finite semidefinite lift. This is the 3×3 case of Fawzi’s Theorem 1 [3, pp.1320–1321], public in its 2019 preprint. His definition allows projected Hermitian LMIs and therefore also excludes real-symmetric lifts. We do not reprove that external theorem.

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Theorem 1. *The body $C_{4,12}$ has no finite semidefinite lift. Consequently (1) fails for some linear functional on $\Lambda^4 \mathbb{R}^{12}$, which may be chosen to have rational coefficients. Some constant closed comass-one four-form on $\mathbb{R}^{12}$ has no norm completion by finitely many four-forms, even at a single point.*

This refutes the original universal degree-one assertion and the general affirmative answer to Question 6.5. The conclusion is existential. We claim no explicit failing form, exact value of its maximum, rational normalized calibration, least dimension or lower-dimensional classification.

2 The Kähler face as a projector body

Identify $V = \mathbb{R}^{12}$ with the underlying real space of $E = \mathbb{C}^6$. Fix an orthonormal real basis $e_1, f_1, \dots, e_6, f_6$ with $Je_j = f_j$, $Jf_j = -e_j$. Set

$$\omega = \sum_{j=1}^6 e_j^* \wedge f_j^*, \quad \Phi = \omega^2/2.$$

The classical Wirtinger inequality gives $\Phi(p) \leq 1$ for $p \in X_{4,12}$, with equality precisely for complex two-planes with their complex orientation [2, 4]. For clarity, the compression of J to a real four-plane has two skew singular values of magnitude at most one. The Pfaffian evaluates Φ as their signed product. Equality forces the compression to preserve every vector's norm, hence J preserves the plane; the positive Pfaffian selects its complex orientation. Conversely, a complex-orthonormal pair a, b gives the real frame a, Ja, b, Jb of value one. Thus

$$F = C_{4,12} \cap \{\Phi = 1\}$$

is exactly the convex hull of these positively oriented complex planes: every positive-weight summand in a convex expression of value one must itself attain one.

Let $W = \Lambda_{\mathbb{C}}^2 E$, of complex dimension 15, with its induced Hermitian metric. In the complexification of V , put $u_j = (e_j - if_j)/\sqrt{2}$, $u_I = u_i \wedge u_j$ for $I = (i, j)$, $i < j$. Define a real-linear map

$$A : \text{Herm}(W) \longrightarrow \Lambda_{\mathbb{R}}^4 V, \quad A(H) = \sum_{I,J} H_{IJ} u_I \wedge \bar{u}_J. \quad (2)$$

The result is real: conjugation transposes H , while swapping exterior factors of degree two has sign $+1$. The 225 vectors $u_I \wedge \bar{u}_J$ form a complex basis of type $(2, 2)$, so A is injective on the 225-dimensional real Hermitian space. With $w_j = e_j - if_j$, the same formula is $A(H) = \frac{1}{4} \sum H_{IJ} w_I \wedge \bar{w}_J$; its real coefficients on the standard Hermitian basis are rational.

For $a \in E$, write $\hat{a} = \sum a_j u_j$. Its underlying real vector satisfies

$$a = (\hat{a} + \bar{\hat{a}})/\sqrt{2}, \quad Ja = i(\hat{a} - \bar{\hat{a}})/\sqrt{2}, \quad a \wedge Ja = -i\hat{a} \wedge \bar{\hat{a}}.$$

For $z = a \wedge b \in W$ this gives

$$A(zz^*) = \hat{a} \wedge \hat{b} \wedge \bar{\hat{a}} \wedge \bar{\hat{b}} = a \wedge Ja \wedge b \wedge Jb. \quad (3)$$

The two factors $-i$ give -1 , and moving $\bar{\hat{a}}$ past $\hat{b}$ cancels that sign. A decomposable unit z has a complex-orthonormal factorization, absorbing its unit phase in a factor. Conversely, every generating plane of F has this form. Therefore

$$F = A(K), \quad K = \text{conv}\{zz^* : z \in W \text{ decomposable, } \|z\| = 1\}. \quad (4)$$

The body K is the established mixed fermionic projector hull [6, Definition 4.2 and Theorem 5.1], rather than a new notion of separability. Equation (4) is a real-linear isomorphism onto the image. Evaluation on the Hermitian basis gives $\Phi(A(H)) = \text{tr } H$: the coefficient on $e_i \wedge f_i \wedge e_j \wedge f_j$ is $H_{(i,j),(i,j)}$.

3 The normalized complex separable-state section

Split $E = U \oplus Z$ orthogonally, using its first and last three complex coordinates. Then

$$W = \Lambda^2 U \oplus B \oplus \Lambda^2 Z, \quad B = U \wedge Z, \quad D = \Lambda^2 U \oplus \Lambda^2 Z.$$

Let P_D be the orthogonal projector onto D , and define $K_0 = \{H \in K : \text{tr}(P_D H) = 0\}$. If $H = \sum_t \lambda_t z_t z_t^*$, with $\lambda_t > 0$ and $\sum_t \lambda_t = 1$, then

$$0 = \text{tr}(P_D H) = \sum_t \lambda_t \|P_D z_t\|^2.$$

Each z_t therefore lies in B . Positivity excludes cancellation and proves the full generator statement, rather than only matrix support.

The unitary map $T : U \otimes_{\mathbb{C}} Z \rightarrow B$ defined by $T(x \otimes y) = x \wedge y$ takes rank-one tensors exactly to nonzero decomposable bivectors in B . This is a specialization of the prior tensor-to-wedge theorem of Vrana–Lévyay [5, §3.1, equation (25)]. Its elementary proof here is short. If $a \wedge b \in B$ is nonzero, the U projections of a, b are dependent, as are their Z projections, because both pure exterior components vanish. Neither projection span is zero, or the wedge would vanish. Taking generators $x \in U, y \in Z$, write $a = \alpha x + \gamma y$, $b = \beta x + \delta y$. Their wedge is $(\alpha\delta - \beta\gamma)x \wedge y$, with nonzero coefficient. The converse is immediate. Orthogonality gives $\|x \wedge y\| = \|x\|\|y\|$; for unit z both factors can be rescaled to unit norm. Under T the projector becomes $xx^* \otimes yy^*$. Hence

$$K_0 \cong \text{Sep}(3, 3). \tag{5}$$

This is an affine section in the original real exterior space. Put

$$\omega_U = \sum_{j=1}^3 e_j^* \wedge f_j^*, \quad \omega_Z = \sum_{j=4}^6 e_j^* \wedge f_j^*, \quad Q = (\omega_U^2 + \omega_Z^2)/2.$$

The same coefficient calculation gives $Q(A(H)) = \text{tr}(P_D H)$, so

$$C_{4,12} \cap \{\Phi = 1, Q = 0\} = A(K_0) \cong \text{Sep}(3, 3). \tag{6}$$

We need $Q \geq 0$ only on F ; it need not be nonnegative on the whole mass ball. The equations are simultaneous, and membership in the mass ball is essential. They alone do not define the separable set.

Affine sections and linear maps preserve finite semidefinite lifts: additional affine equations on a projected point can be imposed on the lift variables. A left inverse of A on its image extends linearly to the ambient space. Thus a lift of $C_{4,12}$ would give a lift of $\text{Sep}(3, 3)$ through (6), contradicting Fawzi. This proves the first assertion of Theorem 1.

4 Supporting affine squares and calibration completion

We give the required compact moment argument explicitly. It specializes the established theta-body and lift framework [7, 8, 9]. Let $X \subset S^{N-1} \subset \mathbb{R}^N$ be nonempty and compact, and set $J_2 = I(X) \cap \mathbb{R}[p]_{\leq 2}$. Choose a finite basis of this vector space. For a symmetric $(N+1)$ -square matrix M , define

$$L_M(1) = M_{00}, \quad L_M(p_i) = M_{0i}, \quad L_M(p_i p_j) = M_{ij}.$$

Consider the spectrahedron and its projection

$$S_X = \{M \succeq 0 : M_{00} = 1, L_M(g) = 0 \ (g \in J_2)\}, \quad T_X = \{(M_{01}, \dots, M_{0N}) : M \in S_X\}.$$

Because $1 - \sum p_i^2 \in J_2$, every feasible matrix has trace two. Positive semidefiniteness bounds all its entries; thus S_X and T_X are compact. Rank-one matrices $(1, p)(1, p)^T$ for $p \in X$ imply $\text{conv}(X) \subseteq T_X$.

If every supporting affine function were an affine SOS on X , then for $h_j(p) = a_j^T(1, p)$ the discrepancy $\delta - \ell - \sum_j h_j^2$ would lie in J_2 . Every feasible M , projecting to x , would satisfy

$$\delta - \ell(x) = \sum_j a_j^T M a_j \geq 0.$$

Intersecting all supporting halfspaces would give $T_X = \text{conv}(X)$, a finite lift. There is no appeal to SDP duality, Slater conditions, or closure of a nonclosed shadow.

Apply this to $X = X_{4,12} \subset S^{494}$, with $N = 495$. Since the mass ball has no finite lift, T_X strictly contains it. Choose $x \in T_X \setminus C_{4,12}$ and strictly separate it by a coefficient vector c :

$$c \cdot x > \max_{p \in X} c \cdot p.$$

The positive gap persists under small perturbations, since x and X are bounded. Choose a rational vector c' in that neighborhood and put $\ell(p) = c' \cdot p$, $\delta = \max_X \ell$. A lifting matrix for x makes $\delta - \ell(x) < 0$, which the preceding SOS calculation forbids. Thus a rational unnormalized failing functional exists. Signed coordinate wedges belong to X and span the exterior space, so nonzero ℓ has $\delta > 0$. The constant form $\varphi = \ell/\delta$ is closed and has comass one, including absolute-value normalization because $X = -X$.

If linear forms ψ_j on the exterior space satisfied the norm completion in Question 6.5, then on unit decomposable vectors

$$\varphi(p)^2 + \sum_j \psi_j(p)^2 = 1 \implies 1 - \varphi(p) = \frac{1}{2}(1 - \varphi(p))^2 + \frac{1}{2} \sum_j \psi_j(p)^2.$$

This affine SOS contradicts the chosen functional. A local completion by differential four-forms would give such a pointwise identity at each point, so none can exist. This proves the remaining assertions.

For clarity on the prior equivalence, the converse at a point is also elementary. If $1 - \varphi(p) = \|a + Bp\|^2$ on X , evaluation at $\pm p$ and spanning give $B^T a = -\varphi/2$ and $\|a\|^2 + \|Bp\|^2 = 1$. At a maximizer p_0 , $Bp_0 = -a$; evaluation at $-p_0$ gives $\|a\|^2 = 1/2$. Projecting onto $a^\perp$ yields $\|\sqrt{2}P_{a^\perp} Bp\|^2 = 1 - \varphi(p)^2$. Homogeneity extends this to all simple vectors. No assertion about choosing smooth local completions from arbitrary pointwise solutions is required.

5 Related results, verification and limits

Nonrepresentability for quadratic Grassmann cones is prior: Bettiol–Kummer–Mendes [12, Corollary 4.2 and Remark 4.3] treat nonnegative homogeneous quadratics in $\text{Sym}^2(\Lambda^k \mathbb{R}^n)$. Blekherman–Smith–Velasco [13, Theorem 1.1] characterize linear SOS for nonnegative projective quadratics. The map $p \mapsto pp^T$ is quadratic and loses orientation; these assertions do not themselves give the needed affine-linear section of the oriented exterior-power mass ball. We make no first Grassmann-related no-shadow claim. Sanyal–Sottile–Sturmfels [14, Theorem 7.6] exclude an unprojected spectrahedron for $G_{3,6}$; their discussion distinguishes projected lifts. That weaker obstruction is not used here.

The verification package includes the complete pinned original proof and coordinate certificate, independently rederived geometry and moment/source audits, bounded priority records and exact arithmetic checkers. The coordinate work checks reality, rank 225, both exposing functionals, the full polynomial projector identity and mixed Plücker/minor relations. Hostile orientation, support, phase and minor cases test fragile boundaries. Finite computations corroborate the elementary bridge; they do not reprove Fawzi, certify the universal conclusion or produce an explicit failing form.

The priority audit used 84 focused queries and 25 primary PDF retrievals, including multiple versions. No inspected body supplied the complete target assembly or negative conclusion, but unindexed or differently formulated work remains possible. This is a bounded no-match finding, not proof of originality or current worldwide openness. The Kähler exposing form itself is a classical positive example, not the existential failing calibration. All lower-dimensional cases remain unaddressed here.

AI tools were used extensively in solving, verifying, source auditing and preparation. This preprint is unrefereed; independent adversarial reviews are AI reviews. Conventional human peer review, journal acceptance and formal proof-assistant certification are not claimed.

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