

Generic Cantor actions of $\mathbb{Z} \wr \mathbb{Z}$ are hyperfinite

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Abstract

We give an affirmative answer to Question 3 in Iyer’s contribution, joint with Shinko, to Oberwolfach Report 2/2025. A dense G_δ of continuous actions of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ on Cantor space has hyperfinite orbit equivalence relation on the entire space. We specialize established Cantor-action lifting principles to obtain a continuous profinite shift height. Finite-width lamp slabs then satisfy the compatible-metric theorem of Conley–Jackson–Marks–Seward–Tucker–Drob. The full preimage of the embedded integers is treated by a separate finite Borel exhaustion. The auxiliary Borel theorem requires the height factor and allows stabilizers.

1 The question and the imported theorems

Let $N = \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}$ and $W = N \rtimes \mathbb{Z}$, with lamp generators a_i and shift t satisfying $ta_it^{-1} = a_{i+1}$. Put $a = a_0$ and let $\chi : W \rightarrow \mathbb{Z}$ be the shift homomorphism. Here $\wr$ denotes the restricted, countable discrete wreath product. Give $\text{Act}(W, 2^\mathbb{N})$ the subspace topology of $\text{Homeo}(2^\mathbb{N})^W$, with compact-open homeomorphism topologies. It is a closed Polish space. Generic means comeager in this full action space. A countable Borel equivalence relation is *hyperfinite* if it is the increasing union of finite Borel equivalence relations.

Theorem 1. *There is a dense G_δ subset of $\text{Act}(W, 2^\mathbb{N})$ such that every action in it has hyperfinite orbit equivalence relation on all of $2^\mathbb{N}$.*

This is the exact whole-space question posed as open in [5, printed p. 115, Question 3]. The locally finite-asymptotic-dimension theorem of Iyer–Shinko [6] does not cover W : it is finitely generated and contains undistorted copies of $\mathbb{Z}^d$ for arbitrarily large d . Our theorem makes no restriction to a conull or comeager subset of the phase space, and does not assert hyperfiniteness for every Borel W -action.

We use three published consequences of Conley–Jackson–Marks–Seward–Tucker–Drob [2], with numbering from their author manuscript. Corollary 5.5 gives finite Borel asymptotic dimension for *any* Borel action of each fixed $\mathbb{Z}^m$. Lemmas 2.2 and 3.1 and the paragraph following Definition 3.2 give its finite-color characterization: for every finite set of group moves there is a Borel coloring with a fixed finite number of colors and uniformly finite monochromatic move components. The number of colors depends on m , not on the move set. Finite Borel covers may be disjointized to obtain colorings.

Theorem 7.3 of [2] says that if proper Borel extended metrics ρ_n on a standard Borel space each have finite Borel asymptotic dimension and

$$\sup\{\rho_{n+1}(x, y) : \rho_n(x, y) < R\} < \infty \quad (n \geq 1, R > 0), \quad (1)$$

then $\bigcup_n E_{\rho_n}$ is hyperfinite, where E_ρ is the relation of finite ρ -distance. No common bound on the dimensions is required. Finally, Corollary 7.5 gives hyperfiniteness for *any* Borel action of a countable group of local polynomial growth; it applies to N . These are deep imported results, rather than claims proved by our diagnostics.

2 Generic compatible heights

Write $\widehat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n!\mathbb{Z}$, with W acting by addition of χ . The generic-height argument below is an explicit specialization of established lifting principles. Akin–Glasner–Weiss [1, Theorem 3.3, equation (3.7)] use conditional compatible lifts for single homeomorphisms. Doucha [3, Proposition 2.12(3)] supplies all-group extension perturbation. Doucha–Melleray–Tsankov [4, Proposition 3.2, Definition A.3, Proposition A.7] give generic finite-extension realization for all Cantor group actions. In their Boolean coding, the finite partition equations below are a specialization of that principle. We include the direct construction to check the prescribed compatibility and the full action-space topology.

Lemma 2. *A dense G_δ of $\text{Act}(W, 2^\mathbb{N})$ admits a continuous equivariant map $p : 2^\mathbb{N} \rightarrow \widehat{\mathbb{Z}}$.*

Proof. Put $\mathcal{A} = \text{Act}(W, 2^\mathbb{N})$. For continuous $c : 2^\mathbb{N} \rightarrow \mathbb{Z}/m\mathbb{Z}$, let $D(c, m)$ be the set of actions α for which

$$c(\alpha(a)x) = c(x), \quad c(\alpha(t)x) = c(x) + 1 \pmod{m}.$$

These equations imply $c(\alpha(g)x) = c(x) + \chi(g)$ for every $g \in W$. The set $D(c, m)$ is clopen: the finite clopen fibers of c give uniform neighborhoods preserving all output labels of both generator maps. There are countably many such c , since every clopen set is a finite union from a countable clopen basis. For $m \mid n$, define the open set

$$U(c; m, n) = (\mathcal{A} \setminus D(c, m)) \cup \bigcup_{d \bmod m=c} D(d, n),$$

where the union ranges over continuous maps $d : 2^\mathbb{N} \rightarrow \mathbb{Z}/n\mathbb{Z}$.

To prove density, fix $\alpha \in D(c, m)$ and an action neighborhood V . The fiber product

$$Y = \{(x, j) \in 2^\mathbb{N} \times \mathbb{Z}/n\mathbb{Z} : c(x) = j \bmod m\}$$

carries the exact continuous action $\gamma(g)(x, j) = (\alpha(g)x, j + \chi(g))$. Choose a finite symmetric set F of group elements controlling V , a finite clopen partition Q of sufficiently small mesh, and a common refinement P of Q , all $\alpha(g)^{-1}Q$ for $g \in F$, and the fibers of c . For each nonempty atom A of P , both A and the inverse image Y_A of A are Cantor spaces: Y_A consists of n/m copies of A . Choose atomwise homeomorphisms $A \rightarrow Y_A$ and combine them into $h : 2^\mathbb{N} \rightarrow Y$. Let $\beta(g) = h^{-1}\gamma(g)h$.

If $h(x) = (u, j)$, then u, x are in the same P -atom. For $g \in F$, $\alpha(g)u$ and $\alpha(g)x$ are in one Q -atom. Also $\beta(g)x$ and $\alpha(g)u$ lie in one P -atom, hence one Q -atom. Thus $\beta(g)$ is uniformly close to $\alpha(g)$ on all of F . Symmetry of F controls inverse maps as well, so $\beta \in V$. The second coordinate d of h is a continuous equivariant residue map modulo n , and $d \bmod m = c$ because P refines the old fibers. This proves density of $U(c; m, n)$; outside $D(c, m)$ membership is automatic.

Intersect these open dense sets over all c, m, n with $m \mid n$. For a fixed action in the resulting dense G_δ , start with $c_1 = 0$ and recursively lift $c_n!$ to $c_{(n+1)!}$. Every map selected in this recursion was already included in the countably many obligations. The compatible residues define continuous equivariant p . Its compact nonempty image is invariant under ± 1 , whose integer translates are dense in $\widehat{\mathbb{Z}}$, so p is onto. A measurable choice of p as the action varies is not required. $\square$

3 The Borel height theorem

Theorem 3. *Suppose W acts Borelly on a standard Borel space X and there is a Borel map $p : X \rightarrow \widehat{\mathbb{Z}}$ with $p(gx) = p(x) + \chi(g)$. Then its entire orbit equivalence relation E_W is hyperfinite.*

No freeness, minimality, invariant probability, or surjectivity of p is assumed in Theorem 3.

Finite-width lamp slabs

For $m \geq 1$, let $r_m(x) \in \{0, \dots, m-1\}$ be the residue of $p(x)$ and $B_m = r_m^{-1}(0)$. This is an N -invariant Borel set. The coordinate map

$$x \longleftrightarrow (z, r) = (t^{-r_m(x)}x, r_m(x)) \in B_m \times \{0, \dots, m-1\}$$

is a Borel bijection with inverse $(z, r) \mapsto t^r z$. Uniqueness uses the prescribed residue, even when lamps have stabilizers.

Let G_m be the undirected graph including all lamp edges $\{x, ax\}$ and the shift edge $\{x, tx\}$ when $r_m(x) < m-1$, with loops omitted. Its degree is at most four. Each distance-at-most- k relation is a finite union of graphs of partial Borel move words, so its path metric ρ_m is a proper Borel extended metric. Proper here means that finite-radius balls are finite. In the displayed coordinates its edges are

$$(z, r) \longleftrightarrow (a_{-r}z, r), \quad (z, r) \longleftrightarrow (z, r+1) \quad (r < m-1),$$

since $t^{-r}at^r = a_{-r}$. The subgroup $K_m = \langle a_0, a_{-1}, \dots, a_{-(m-1)} \rangle \cong \mathbb{Z}^m$ acts on B_m .

Lemma 4. *Each ρ_m has finite Borel asymptotic dimension.*

Proof. By [2, Corollary 5.5] and its finite-move characterization, there is a fixed finite color count $D_m + 1$ such that, for each $R \geq 1$, B_m has a Borel coloring whose monochromatic components under K_m words of length at most $\lceil R \rceil$ have at most $C_R < \infty$ points. Lift the color of z to every (z, r) . A slab path of length at most R changes its base by at most $\lceil R \rceil$ lamp letters. Hence a monochromatic ρ_m -distance-at-most- R chain projects into one monochromatic base component. Connecting paths need not themselves be monochromatic; only their endpoints are used. The lifted component has at most mC_R points, and its diameter is at most $R(mC_R - 1)$ by a simple path in the threshold graph. For $0 < R < 1$ the components are singletons. This is the finite-color characterization of finite Borel asymptotic dimension in [2, Lemma 3.1]. No bound uniform in m is asserted. $\square$

Compatible exhaustion on the regular part

If $m \mid M$, a forbidden residue $M-1$ reduces to the forbidden residue $m-1$. Thus $G_m \subseteq G_M$ and $\rho_M \leq \rho_m$ on all pairs of finite old distance. In particular factorial widths satisfy (1), with bound R .

The embedded integers form a countable Borel subset of $\widehat{\mathbb{Z}}$. Put $X_{\text{exc}} = p^{-1}(\mathbb{Z})$ and $X_{\text{reg}} = X \setminus X_{\text{exc}}$; both are Borel and W -invariant. On X_{reg} , every forward shift edge eventually appears in a factorial slab: a point with $r_n!(x) = n! - 1$ for every n would have $p(x) = -1$, contrary to membership in this set. Once admitted, an edge stays admitted. Lamp edges are always present. Each finite generator word therefore fits in one sufficiently late slab, giving

$$E_W \upharpoonright X_{\text{reg}} = \bigcup_n E_{\rho_{n!} \upharpoonright X_{\text{reg}}}.$$

Invariance means that restricting the graphs removes no intervening path. Lemma 4 and [2, Theorem 7.3] imply hyperfiniteness here. This invokes the compatible-metric theorem, not a general increasing-union principle for hyperfinite relations.

The full exceptional preimage

For $x \in X_{\text{exc}}$, let $h(x)$ be the unique integer equal to $p(x)$, and put $B = p^{-1}(0)$ and $q(x) = t^{-h(x)}x \in B$. Both h and q are Borel, and N preserves B . For $y = gx$, the element $t^{-h(y)}gt^{h(x)}$ has zero shift character and lies in N . Conversely an N -move between normalized points conjugates back to a W -move. Therefore

$$x E_W y \iff q(x) E_N q(y).$$

Every finitely generated subgroup of N has polynomial growth, so [2, Corollary 7.5] supplies increasing finite Borel relations F_j on B exhausting its N orbit relation. On X_{exc} define

$$x H_j y \iff x = y \text{ or } (|h(x)|, |h(y)| \leq j \text{ and } q(x) F_j q(y)).$$

Inside the band this is a pullback of F_j ; outside it all classes are singletons. For each $z \in B$ and $k \in \mathbb{Z}$ there is exactly one point with $q(x) = z$ and $h(x) = k$, namely $t^k z$. Thus band classes are finite, of size at most $(2j + 1)|[z]_{F_j}|$. The relations H_j are Borel, increase, and exhaust the full exceptional orbit relation. The exceptional set is not assumed countable or negligible; pulling back F_j without the height band would not give finite classes.

Take the disjoint union of finite increasing exhaustions of the two invariant pieces. This proves Theorem 3 on all of X . Lemma 2 then proves Theorem 1.

4 Verification and publication status

The accompanying package preserves the original proof and reviews, adds fresh independent mathematical and dated priority audits, and supplies portable finite diagnostics and integrity checks. Finite computations check group identities, nonfree slab models, divisor cuts, projections, and bounded-height exhaustions; they do not prove the infinite theorem. The proof above relies on the precise published inputs stated in Section 1.

The bounded priority audit of October 13, 2026 found no exact earlier resolution in its inspected corpus and identified the lifting predecessors credited above. It does not certify historical firstness or worldwide current openness. This note answers the historically posed exact question.

OpenAI tools were used extensively in developing and checking the argument, drafting the note, writing diagnostics, and conducting independent adversarial AI reviews. This is an unrefereed preprint. No completed external human peer review or formal proof-assistant verification is claimed.

References

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