# An $n = 8$ counterexample to Karpenko's gamma-filtration conjecture

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17 October 2026

**Abstract**

Karpenko's gamma-filtration nonmembership conjecture is false at $n = 8$. We spell out this consequence of his prior results: the later analytic isotropy theorem for Spin(18), combined with the original quadric-transfer argument and filtration theorem, gives $e_8 \in \gamma^5 K_0(X_K)$ in the conjecture's precise maximal-Clifford-index and Severi–Brauer function-field setup. An iterated Laurent-series field provides an explicit characteristic-zero model. We also supply exact arithmetic witnesses reproducing the previously published class valuation underlying the earlier computational proof. This is a credited exposition and verification of a literature-derived negative corollary; no firstness claim is made for the corollary or its particular arithmetic witnesses.

## 1 The original statement and the negative corollary

Let $\varphi/F$ be a nondegenerate quadratic form of dimension $2n + 1$, with even Clifford algebra $A = C_0(\varphi)$ division of degree $2^n$. Let $X = \mathrm{OG}_n(\varphi)$ parametrize maximal totally isotropic vector subspaces, let $S = \mathrm{SB}(A)$ be the ordinary Severi–Brauer variety, and put $K = F(S)$. We use integral $K_0$ throughout. For a smooth variety $V$, $T^i K_0(V)$ is generated by coherent sheaves supported in codimension at least $i$; $\gamma^i K_0(V)$ is the usual gamma filtration of its augmented lambda ring.

Restriction identifies $K_0(X_K)$ with the split Grothendieck ring [4, §4]. Thus the structure-sheaf class of the split special Schubert variety of codimension $n$ specifies a class $e_n \in K_0(X_K)$. This definition does not assert that the corresponding Schubert subvariety is defined over $K$. Karpenko's conjecture [3, Conjecture 3], also [4, Conjecture 4.7], asserts

$$e_n \notin \gamma^{\,n+1-\lfloor (n+1)/2 \rfloor} K_0(X_K) \qquad (n \geq 8). \tag{1}$$

**Theorem 1.** *Take*

$$F = \mathbb{C}((t_1))\cdots((t_{16})), \qquad \varphi = \langle 1, -t_1, \ldots, -t_{16} \rangle.$$

*Then $C_0(\varphi)$ is division of index 256. With $X, S, K$ as above,*

$$e_8 \in \gamma^5 K_0(X_K).$$

*Consequently* (1) *fails at $n = 8$.*

The proof combines published results rather than asserting a new Spin index theorem. The required generic fourth-Grassmannian index 8 occurs in author versions dated 2021 of [5], is summarized in [7], and is fully published in [5]. Karpenko's later analytic argument [6, Theorem 4.1 and Corollary 4.4] provides the input used below without the earlier computer calculation. The transfer from that input to the exact forbidden membership is already given in [4, following Conjecture 4.7]. We make its application and field identifications explicit. Priority for an earlier printed statement of this combined conclusion has not been established; we do not assert that the problem remained open until this note. Nor do we determine exact gamma depth, an optimal replacement conjecture, or failure at every $n \geq 8$.

## 2 The Clifford model and the isotropy input

*Verification of the model.* Give $F$ its composite rank-sixteen valuation, with value group $\Gamma \simeq \mathbb{Z}^{16}$ and the sixteen values $v(t_i)$ a basis. If $e_0, e_1, \ldots, e_{16}$ are Clifford generators for $\varphi$, then $u_i = e_0 e_i$ satisfy

$$u_i^2 = t_i, \qquad u_i u_j = -u_j u_i \quad (i \neq j).$$

The ordered squarefree monomials $u_I$ form an $F$-basis of $C_0(\varphi)$. Assign $u_I$ the value $\frac{1}{2}\sum_{i\in I} v(t_i)$ in $\frac{1}{2}\Gamma$. Their values lie in distinct cosets modulo $\Gamma$. Every nonzero linear combination $\sum_I a_I u_I$ therefore has a unique term of least value. Multiplication of basis monomials respects these values and has nonzero coefficient. The product of the two least terms is the unique least term of a product, so two nonzero elements cannot have zero product. The even Clifford algebra is central simple of dimension $2^{16}$; absence of zero divisors makes it division, of degree $2^8$. □

Set

$$p = t_1 \cdots t_{16}, \qquad \psi = \varphi \perp \langle -p \rangle.$$

The signed discriminant of $\psi$ is $p^2$, hence trivial. The standard codimension-one Clifford and Grassmannian identifications [4, §1] give

$$C^+(\psi) \simeq C_0(\varphi) = A, \qquad X \simeq \mathrm{OG}_9^+(\psi),$$

where either maximal component can be used. Since $K = F(\mathrm{SB}(A))$ splits $A$, the form $q = \psi_K$ has trivial discriminant and Clifford invariant. No assertion that $q$ itself is hyperbolic is involved.

For a projective variety $W$ over a field $k$, its index is the positive generator of the ideal of degrees of zero-cycles:

$$\mathrm{ind}(W) = \gcd\{[k(w) : k] : w \text{ a closed point of } W\}.$$

**Lemma 2** (Karpenko's isotropy bound)**.** *If $q/k$ is an 18-dimensional nondegenerate quadratic form in characteristic zero with trivial discriminant and Clifford invariant, then* $\mathrm{ind}\,\mathrm{OG}_4(q)$ *divides* 8.

*Proof.* The universal consequence of [6, Theorem 4.1 and Corollary 4.4] states that $q$ acquires Witt index at least 5 over a finite extension $L/k$ whose degree is not divisible by 16. In particular $\mathrm{OG}_4(q)(L) \neq \varnothing$, so its index divides $[L : k]$. A diagonalization of $q$ and adjoining square roots gives a splitting extension of 2-power degree. Its Grassmannian therefore has 2-power index. These two divisibilities imply the assertion. □

The generic value in [6] is exactly 8; only divisibility by 8 is used for our particular model. Alternatively the generic result of [5] gives the same bound by generic-to-arbitrary divisibility [2, Proposition 2.1]: the generic top-degree Chow image is contained in the arbitrary one, so the arbitrary index divides the generic index. This does not identify two different meanings of "generic."

## 3 Transfer to the original gamma filtration

*Proof of Theorem 1.* Write $Y$ for the projective quadric of $q = \psi_K$; thus $\dim Y = 16$. On a split quadric write $\ell_j$ for the structure-sheaf class of a projective $j$-space, and put $h = 1 - [\mathcal{O}_Y(-1)]$. The split-Clifford quadric $K$-theory identification used in [4, §4] gives these classes in $K_0(Y)$ and makes restriction to a splitting field injective. They can be virtual classes: in particular $\ell_0$ does not assert a $K$-rational point or membership in $T^{16}K_0(Y)$.

Let $w$ be a closed point of $\mathrm{OG}_4(q)$, with residue field $L$ and degree $d$. The corresponding isotropic vector 4-space gives a projective 3-space on $Y_L$. Its class belongs to $T^{13}K_0(Y_L)$. Finite pushforward preserves support codimension. After restriction to a common splitting field it takes this class to $d\ell_3$. Injectivity therefore gives

$$d\ell_3 \in T^{13}K_0(Y).$$

By Lemma 2, integral combinations of such degrees equal 8, and hence $8\ell_3 \in T^{13}K_0(Y)$. An extension of degree exactly 8 is unnecessary.

The integral quadric identity [4, proof following Conjecture 4.7] specializes to

$$8\ell_3 = \ell_0 + h^{15} + 2h^{14} + 4h^{13}. \tag{2}$$

For instance it follows by iterating $2\ell_j = \ell_{j-1} + h^{16-j}$ for $j = 1, 2, 3$. The hyperplane terms lie in $T^{13}$, so (2) gives $\ell_0 \in T^{13}K_0(Y)$, without division by 8.

Use the even maximal component $X_K \simeq \mathrm{OG}_9^+(q)$ and its rank-nine tautological bundle $U$. The incidence variety of an isotropic line inside $U$ has projections

$$Y \xleftarrow{g} Z = \mathbf{P}(U) \xrightarrow{f} X_K.$$

Here $f$ is a projective bundle of relative dimension 8, and $g$ is smooth. The dimensions are $\dim X_K = 36$, $\dim Z = 44$, and $\dim Y = 16$; the relative dimension of $g$ is 28. Flat pullback by $g$ preserves support codimension, and proper pushforward by $f$ lowers it by at most 8. The split Schubert identity, valid in $K$-theory and transported by restriction, is $e_8 = f_*g^*(\ell_0)$ [4, §4, the $K$-theoretic version of (1.2)]. Consequently

$$e_8 \in T^{13-8}K_0(X_K) = T^5K_0(X_K).$$

Finally [4, Theorem 4.3] applies to our original odd form: $A$ is division of maximal degree over $F$, and $K$ is precisely the function field of its ordinary Severi–Brauer variety. It gives $T^iK_0(X_K) = \gamma^iK_0(X_K)$ over this $K$ itself. Thus $e_8 \in \gamma^5K_0(X_K)$, whereas the term forbidden by (1) at $n = 8$ is $8 + 1 - \lfloor 9/2 \rfloor = 5$. □

The equality of filtrations in the last step is essential. Transporting the underlying $K$-ring to a splitting field transports the gamma filtration but need not make the topological filtration strict. In particular, the split geometric codimension of the special Schubert variety does not by itself settle this membership question.

## 4 An exact certificate for the earlier finite calculation

This section independently reproduces a published fact from [5, Appendix A]; it is additional verification, not a dependency of the analytic proof above. Let $V = \mathrm{IG}(5, 16) = \mathrm{Sp}_{16}/P_5$, of

dimension 45, and let $T$ be its rank-five tautological bundle. Write $x_1, \dots, x_5$ for formal roots of $T^*$, $\pm y_1, \pm y_2, \pm y_3$ for those of $Q = T^\perp/T$, and $u = \sum_i x_i$. The class in question is

$$a = \prod_{\epsilon \in \{\pm 1\}^3} (u + \epsilon_1 y_1 + \epsilon_2 y_2 + \epsilon_3 y_3) = 2^8 \widetilde{z}. \tag{3}$$

Here $\widetilde{z}$ denotes the source's orbit product, interpreted in the rational Chow ring for this identity. There are eight doubled orbit factors. The prose in the inspected author and journal versions of Appendix A prints $2^3\widetilde{z}$; the product and §§4–5 require the normalization $2^8$ in (3). We record this discrepancy without asserting an official erratum or first identification. Simultaneously negating all roots leaves this degree-eight product unchanged, so using $T^*$ is consistent.

Put $e_i = c_i(T^*)$, and let $h_j(x)$ be the complete homogeneous symmetric polynomial. The exact sequences for $T \subset T^\perp$ give $c(T)c(T^*)c(Q) = 1$, hence

$$h_j(x_1^2, \dots, x_5^2) = 0 \quad (j \geq 4). \tag{4}$$

Indeed these are the Chern classes $c_{2j}(Q)$ of a rank-six bundle. If $q_i = e_i(y_1^2, y_2^2, y_3^2)$, expansion of (3) is

$$\begin{aligned} a = {} & u^8 - 4q_1 u^6 + (6q_1^2 - 8q_2)u^4 \\ & + (-4q_1^3 + 16q_1 q_2 - 64q_3)u^2 + (q_1^2 - 4q_2)^2. \end{aligned}$$

Substitute $q_i = (-1)^i h_i(x^2)$ and call the resulting polynomial $A$. Direct integral polynomial expansion gives $A = 128b + 64h_4(x^2)$, where

$$\begin{aligned} b = {} & -e_4^2 - e_3e_5 + 2e_2e_3^2 + 5e_2^2e_4 - 2e_2^4 + 3e_1e_3e_4 + 4e_1e_2e_5 \\ & - 9e_1e_2^2e_3 - 2e_1^2e_3^2 - 6e_1^2e_2e_4 + 7e_1^2e_2^3 - e_1^3e_5 \\ & + 7e_1^3e_2e_3 + e_1^4e_4 - 5e_1^4e_2^2 - e_1^5e_3 + e_1^6e_2. \end{aligned}$$

Thus $a = 128b$ in $\mathrm{CH}(V)$. A second polynomial identity is

$$b \equiv (e_1e_3 + e_1^2e_2)h_4(x) + e_3h_5(x) + e_2h_6(x) + e_1h_7(x) + h_8(x) \pmod 2.$$

Squaring modulo 2 gives $h_j(x)^2 = h_j(x^2)$, and (4) gives $b^2 = 0$ in $\mathrm{CH}(V)/2$. Therefore $a^3 = 2^{21}b^3$ is divisible by $2^{22}$ integrally.

The complementary degree is the exact integer

$$\int_V a^3 c_2(T^*)^9 c_3(T^*) = 14668659594398531584 = 2^{22} \cdot 3497280977821. \tag{5}$$

Here the complement has codimension $21 = 9 \cdot 2 + 3$, as required. For an integer-only verification, $V$ is the zero locus of the regular section of $\Lambda^2 T^*$ induced by a fixed nondegenerate alternating form on $\mathrm{Gr}(5, 16)$, with class $\prod_{i<j}(x_i + x_j)$. The usual Grassmannian coefficient formula [1] therefore evaluates the normalized degree as

$$[x_1^{15}x_2^{14}x_3^{13}x_4^{12}x_5^{11}] \left( b^3 e_2^9 e_3 \prod_{i<j}(x_i + x_j)(x_i - x_j) \right) = 6994561955642.$$

Multiplication by $128^3$ gives (5). The archive evaluates this by integral vertical-strip Pieri multiplication and, separately, root-monomial coefficient extraction; exact fixed-point localization provides further checks. These are established Schubert and Gysin methods. The classes $c_2(T^*)$ and $c_3(T^*)$ here are dual-tautological Chern classes, with no change to quotient-bundle one-row conventions. Since 3497280977821 is odd, the pairing prohibits divisibility of $a^3$ by $2^{23}$. Together with the lower bound above, this reproduces exact class valuation $\nu_2(a^3) = 22$ in the free integral Chow group. The valuation is prior work; priority for the particular pairing and displayed identities remains unestablished.

## 5 Verification and research disclosure

The supporting verification archive contains self-contained exact integer and finite-field checkers, source/version receipts, and mathematical and priority review records. Independently implemented routes reproduce both (5) and the mod-2 coefficient $[z^8x_7^4x_8^4]\prod_{I\subseteq\{5,6,7,8\}}(z+\sum_{i\in I}x_i)=1$ used in [6]. Recorded executions use ordinary Python and its optimization modes `-O` and `-OO`. The original Maple program and its full printed Schubert expansion were not replayed. These finite checks do not reprove the cited Clifford, Chow, isotropy, filtration or Gysin theorems. Third-party source bodies are omitted from the archive; the inspected final author versions and bibliographic records are distinguished in its source ledgers.

AI tools were used extensively in deriving and checking the argument, literature retrieval and priority assessment, exact computation, drafting, and internal adversarial review. This preprint is unrefereed; it has not undergone external human peer review or proof-assistant certification. The mathematical claims and their source dependencies are presented for independent verification.

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