Full stress stratifications determine a labelled graph

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Abstract

For a simple labelled graph, consider the full configuration space, including vertex collisions, partitioned into connected components of coordinate-sign equivalence classes of self-stress spaces. We prove that two graphs induce identical partitions precisely when their edge sets agree. More explicitly, an edge is recovered by asking whether its collision diagonal is a union of strata. This answers the literal equality formulation of Karpenkov's Problem 4. The proof combines the known singleton-support collision criterion with an explicit positive diagonal transport along a collinear path. The comparison convention is essential: nonisomorphic trees can have ambient-linearly equivalent stratifications.

1 The full partition and its reconstruction

Fix integers $n, d \geq 1$. A graph below is simple and has the common labelled vertex set $[n]$; unused vertices are isolated. Its configuration space is $X = (\mathbb{R}^d)^n$. For $P = (p_1, \ldots, p_n)$ and a graph G, put

$$W_G(P) = \left\{ w \in \mathbb{R}^{E(G)} : \sum_{j:\{i,j\}\in E(G)} w_{ij}(p_j - p_i) = 0 \text{ for every } i \right\}.$$

There is one real coordinate per unordered edge, so $w_{ij} = w_{ji}$. Two configurations for the *same* graph are equivalent when a homeomorphism between their stress spaces preserves the sign of every edge coordinate, including zero. Let $\mathcal{S}_d(G)$ be the partition of X into connected components of these equivalence classes. We retain the entire Cartesian space: collisions, self-intersections and deficient affine spans are allowed. These are the full-space conventions of [1, 2].

Equality of partitions means equality as collections of subsets of this *fixed labelled space*. We do not directly compare stress spaces belonging to different graphs. Write $\Delta_{ij} = \{P \in X : p_i = p_j\}$ for a collision diagonal.

Theorem 1. *For every simple labelled graph G and every $i \neq j$,*

$$\{i, j\} \in E(G) \quad \Longleftrightarrow \quad \Delta_{ij} \text{ is a union of strata of } \mathcal{S}_d(G).$$

Consequently, for simple graphs G, H on $[n]$,

$$\mathcal{S}_d(G) = \mathcal{S}_d(H) \quad \Longleftrightarrow \quad E(G) = E(H).$$

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Proof. If $e = \{i, j\}$ is an edge, the stress positive only on e belongs to $W_G(P)$ exactly when $p_i = p_j$: its only possibly nonzero forces are $p_j - p_i$ and $p_i - p_j$. Thus a coordinate-sign-preserving homeomorphism of fibres cannot identify a collapsed and a noncollapsed placement of e. The collision diagonal is therefore a union of whole equivalence classes, hence of strata. This elementary collision detector is already recorded in Panina's Lemma 3.1 (preprint Section 3, Lemma 4) [3]; we claim no novelty for it.

Conversely, suppose $\{i, j\}$ is not an edge of G. Choose $a \in \mathbb{R}^d \setminus \{0\}$ and, for $-1/2 \le t \le 1/2$, set

$$p_k(t) = x_k(t)a, \qquad x_i(t) = t, \quad x_j(t) = 0, \quad x_k(t) = k + 1 \ \ (k \notin \{i, j\}).$$

The only pair that can collide on this path is $\{i, j\}$, so no G-edge collapses. Orient the edges arbitrarily and let B_G be the vertex-edge incidence matrix, with $+1$ at the initial endpoint and -1 at the final one. For an oriented edge (u, v) put $D_t = \operatorname{diag}(x_v(t) - x_u(t))$. Since $a \neq 0$,

$$W_G(P(t)) = \ker(B_G D_t), \qquad T_t = D_t^{-1} D_0, \qquad B_G D_t T_t = B_G D_0.$$

Every entry of D_t stays nonzero with constant sign. Hence T_t is a positive diagonal linear homeomorphism from $W_G(P(0))$ onto $W_G(P(t))$; its inverse is $D_0^{-1} D_t$. The construction is independent of edge orientation and preserves zero coordinates as well as both nonzero signs. For an edgeless graph it is the unique map between zero stress spaces.

The entire continuous path thus lies in one fibre-equivalence class. Its connected image lies in one connected component of that class, hence in one stratum. But $P(0) \in \Delta_{ij}$ and $P(1/4) \notin \Delta_{ij}$. The diagonal cannot be a union of whole strata. This proves the equivalence. Identical partitions now recover identical edge sets; the reverse implication is immediate. For $n = 1$ there is only the empty simple graph. □

The collinear path is a witness *inside* the full configuration space; the theorem is not a classification restricted to collinear placements. Stress-space dimension alone does not supply the argument: it is the labelled coordinate support that detects a collapsed edge.

Corollary 2. *A specified permutation of vertex labels carries $\mathcal{S}_d(G)$ onto $\mathcal{S}_d(H)$ exactly when it carries G onto H.*

Proof. Reindexing points and edge weights carries the equilibrium equations for G to those for the relabelled graph. Apply Theorem 1. □

2 The original question and the comparison convention

Karpenkov's Problem 4 [2, p. 22] asks which subgraphs of K_n define the same stratifications. Its preceding definitions use connected strata in the full labelled Cartesian configuration space. Under literal equality of these defined partitions, Theorem 1 gives the complete answer: each equality class contains exactly one labelled graph. The source does not separately define a cross-graph abstract or ambient equivalence. We do not infer an author's unwritten intention or claim to classify such different comparisons, generic-only spaces, or collision-free spaces.

The distinction has a concrete mathematical effect. Let G be the four-vertex path with edges $12, 23, 34$, and H the star with edges $12, 13, 14$. For any forest, leaf elimination shows that stresses are supported exactly on its collapsed edges; those coordinates are arbitrary and independent. The invertible linear transformation

$$q_1 = p_1, \quad q_2 = p_2, \quad q_3 = p_1 + p_3 - p_2, \quad q_4 = p_1 + p_4 - p_3$$

turns the path's three edge differences into the star's three edge differences. It therefore maps their full stress-equivalence classes, and their connected components, bijectively under that edge correspondence. The graphs have different degree multisets, so they are nonisomorphic, although their stratifications are ambient-linearly equivalent. Their partitions are unequal in the fixed labelled space by the theorem.

Prior credit and verification. The stress-space convention comes from [1, 2]; the singleton-support detector is credited above. Collinear order invariance of the stress matroid is also noted in [3, Appendix A.3; preprint v4, Section 4.1]. We give the incidence transport explicitly for a self-contained proof and claim no novelty for these ingredients. The conclusion compared here is reconstruction from the full fixed-label partition. Panina's affine-quotient and noncollapsed-edge framework is not substituted for the full space used here. The supplement records source and priority checks, independent adversarial derivations, and reproducible exact-rational calculations. Finite tests corroborate the identities and boundary cases; they do not prove the universal or connectedness statements and provide no formal proof-assistant certificate. No first-priority claim is made for the cited ingredients.

AI use and review status. AI tools were used extensively in developing and checking the arguments, auditing sources, and preparing this manuscript and its supporting materials. This is an unrefereed research note. The independent adversarial reviews described in the supplement are AI reviews; conventional human peer review or journal acceptance is not claimed.

References

[1] F. Doray, O. Karpenkov and J. Schepers, *Geometry of configuration spaces of tensegrities*, Discrete Comput. Geom. **43** (2010), 436–466. doi:10.1007/s00454-009-9229-4. Author preprint: arXiv:0806.4976v2.

[2] O. Karpenkov, *Open Problems on Configuration Spaces of Tensegrities*, Arnold Math. J. **4** (2018), 19–25. doi:10.1007/s40598-018-0080-7.

[3] G. Panina, *A universality theorem for stressable graphs in the plane*, Ars Math. Contemp. **18** (2020), 137–148. doi:10.26493/1855-3974.641.e06. Published Lemma 3.1 and Appendix A.3; author version: arXiv:1902.07212v4, Section 3, Lemma 4.