

Free spanning-forest degree can decrease under squaring

Alec Kriebel

Independent researcher

ORCID: 0009-0001-9320-500X

22 October 2026

Preprint, version 1.0.0

Abstract

The expected root degree of the free uniform spanning forest need not be invariant under squaring an infinite connected locally finite simple vertex-transitive graph. For the binary grandparent graph G , of degree 8, two explicit free-exhaustion reductions give $\mathbb{E} \deg_{\text{FUSF}(G)}(o) = (24 - \sqrt{2})/7 > 3.22$. For its simple square G^2 , of degree 40, a finite rational-current certificate proves $\mathbb{E} \deg_{\text{FUSF}(G^2)}(o) \leq 3208526010913748780/10^{18} < 3.21$. All currents, divergences and energies are verified with integer arithmetic; the passage to the infinite free forest is justified by finite-support flows and free exhaustion. With the explicitly adopted normalization $b_F(X) = \frac{1}{2} \mathbb{E} \deg_{\text{FUSF}(X)}(o) - 1$, this gives $b_F(G) - b_F(G^2) > 1/200$, a negative answer to the unrestricted vertex-transitive assertion in Abért's Question 24. The graph and electrical/forest principles are established background; no new general forest mechanism or exhaustive priority claim is made.

1 The question and the precise convention

For a simple graph X , its simple square X^2 has the same vertices, with distinct vertices adjacent when their X -distance is at most two. Every edge below has unit conductance. The free uniform spanning forest, abbreviated FUSF, is the weak limit of uniform spanning trees on finite connected induced subgraphs exhausting X , with no boundary vertices identified [2, §5]. For vertex-transitive X , define

$$b_F(X) = \frac{1}{2} \mathbb{E} \deg_{\text{FUSF}(X)}(o) - 1. \quad (1)$$

The choice of o does not matter.

Abért's *Some questions*, dated 2 November 2010, printed p. 4, Question 24 [1], asks square invariance for the first L^2 Betti number obtained from expected free-forest degree for an *arbitrary* vertex-transitive graph. The original does not print a normalization formula. We state (1) explicitly: it is the usual forest-derived normalization matching $\mathbb{E} \deg_{\text{FUSF}} = 2 + 2\beta_1^{(2)}$ on Cayley graphs [4, p. 1]. Our directly proved assertion is unequal expected degrees, and the answer about b_F uses (1). We do not identify a different nonunimodular L^2 invariant with it. No unimodularity hypothesis is present in the inspected question; its adjacent comment already affirms the Cayley case.

Theorem 1. *Let T be the 3-regular tree with a distinguished end, let $p(v)$ be the neighbor toward that end, and let G have the unordered edges $\{v, p(v)\}$ and $\{v, p^2(v)\}$. Then G is an infinite connected locally finite simple vertex-transitive graph of degree 8, with nonunimodular full automorphism group. Its square $H = G^2$ is vertex-transitive of degree 40, and*

$$\mathbb{E} \deg_{\text{FUSF}(G)}(o) = \frac{24 - \sqrt{2}}{7} > \frac{161}{50}, \quad (2)$$

$$\mathbb{E} \deg_{\text{FUSF}(H)}(o) \leq \frac{3208526010913748780}{10^{18}} < \frac{321}{100}. \quad (3)$$

Consequently $b_F(G) - b_F(H) > 1/200$.

The grandparent family is prior; Tang studies its FUSF connectivity and branching [5, §5.2]. The contribution here is the specific expected-degree comparison, using the exact original value and a finite certificate for the square. The known Cayley case and unimodular framework are not refuted. We make no assertion about a new general electrical method.

2 The graph and the electrical facts used

The end stabilizer in $\text{Aut}(T)$ acts transitively and preserves G , proving transitivity; G contains T . At any vertex there are three incident tree edges and five incident grandparent edges, with no repetitions.

We also verify nonunimodularity for the *full* group. A tree edge $xp(x)$ lies in exactly three triangles of G , through $p^2(x)$ and the two children of x . A grandparent edge $xp^2(x)$ lies in exactly one, through $p(x)$. Thus G intrinsically determines the tree edges. On the three recovered tree-neighbors of x , the induced G -graph is $K_{1,2}$, whose unique center is $p(x)$. Every automorphism therefore preserves p and the distinguished end. Conversely an end-preserving tree automorphism preserves G . The stabilizer of x fixes $p(x)$, while the stabilizer of $p(x)$ has an orbit of size two on its children, including x . The paired suborbit sizes 1, 2 violate the usual unimodularity criterion [5, §2.3]. A nonunimodular transitive subgroup alone would not have sufficed for this full-group conclusion.

For finite connected unit-conductance networks, resistance is the minimum of

$$\mathcal{E}(\theta) = \sum_{\{x,y\}} \theta(x,y)^2$$

over unit flows between the specified terminals; sum once per *unoriented* edge. We use Thomson's principle, Rayleigh monotonicity and Kirchhoff's edge-marginal identity [2, §4, Theorem 4.1]. Define free resistance by a finite free induced exhaustion:

$$R_X^F(x, y) = \lim_n R_{X_n}(x, y).$$

The limit decreases and is exhaustion-independent. The finite identity and the free weak limit give, for every edge $e = \{x, y\}$,

$$\mathbb{P}(e \in \text{FUSF}(X)) = R_X^F(x, y). \quad (4)$$

See also [2, Proposition 7.1 and Theorem 7.8]. Connected edge-exhaustions give the same limit: they are eventually sandwiched between any fixed finite induced ball and some larger ball, so Rayleigh monotonicity squeezes the resistance limits. These facts concern free boundaries, not wired resistance or the unconstrained minimum among infinite square-summable flows.

3 First calculation: branches and a diagonal free exhaustion

For a vertex u , write $a = p(u)$ and $b = p^2(u)$. Let $B_m(u; a, b)$ contain u , its descendants through m generations, and terminals a, b . Include every grandparent-graph edge incident to an interior vertex with both endpoints in this set, and omit the edge ab . An untruncated descendant branch meets the rest of G only at a, b . The truncation deletes external edges; it makes no identifications. Different child branches have disjoint interiors.

Let κ_m be the terminal conductance of B_m . The case $m = 0$ is the two-unit-edge path $a - u - b$. Replacing each of u 's two child branches by its terminal conductance therefore gives the exact recurrence

$$\kappa_0 = \frac{1}{2}, \quad \kappa_{m+1} = \frac{1 + 2\kappa_m}{2 + 2\kappa_m}. \quad (5)$$

Indeed the ua conductance becomes $1 + 2\kappa_m$, in series with the unit edge ub . The map in (5) is increasing, starts the sequence increasing from $1/2$, and is bounded above by 1. Its unique positive fixed point is

$$\kappa = 1/\sqrt{2}, \quad A = 1 + 2\kappa = 1 + \sqrt{2}, \quad \kappa = \frac{A}{1 + A}.$$

Put $x_i = p^i(o)$. For $m \geq 0, n \geq 2$, form the following finite connected subgraph $L_{m,n}$: the spine $x_0, \dots, x_n$ with all adjacent and skip-one edges; both depth- m child branches at x_0 , with terminals x_0, x_1 ; and the one depth- m off-spine child branch at each x_i , $1 \leq i < n$, with terminals x_i, x_{i+1} . All original distinct edges retain conductance one. These graphs increase with each parameter. Every end-ray from a fixed vertex merges with the spine at a finite place; its off-spine depth is finite. Thus $L_{n,n}$ eventually contains every vertex and every edge of G . The temporarily omitted branches at the top x_n appear at later stages. This is a genuine free edge-exhaustion.

Eliminate the attached branches. The conductance on x_0x_1 is $A_m = 1 + 2\kappa_m$; on each higher adjacent pair it is $C_m = 1 + \kappa_m$; each skip-one edge still has conductance one. For the upper suffix $x_1, \dots, x_n$, its terminal conductance is

$$B_{m,n} = T_{C_m}^{n-2}(C_m), \quad T_C(t) = C + \frac{t}{1+t}. \quad (6)$$

This follows by eliminating the top triangular cell successively: a direct C is parallel to a unit edge in series with the remaining upper conductance.

For $t \geq C_m \geq 1$, $T'_{C_m}(t) \leq 1/4$. Here C_m stays in a compact interval and the positive fixed point of T_C , the positive root of $B^2 - CB - C = 0$, depends continuously on C . Hence the convergence of the iterates in (6) is geometrically uniform in m . As $m = n \rightarrow \infty$, $C_m \rightarrow C = 1 + \kappa$ and $B_{n,n} \rightarrow A$, since $A = C + A/(1 + A)$. This justifies the actual diagonal limit; no unsupported interchange of two infinite limits is made.

The reduced three-terminal network has conductances $A_m, B_{m,n}, 1$ on x_0x_1, x_1x_2, x_0x_2 . Thus

$$R_{L_{m,n}}(x_0, x_1) = \left(A_m + \frac{B_{m,n}}{1 + B_{m,n}} \right)^{-1}, \quad (7)$$

$$R_{L_{m,n}}(x_0, x_2) = \left(1 + \frac{A_m B_{m,n}}{A_m + B_{m,n}} \right)^{-1}. \quad (8)$$

Along the free exhaustion, these converge respectively to

$$R_{\text{tree}} = \frac{1}{1 + 3\kappa} = \frac{3\sqrt{2} - 2}{7}, \quad R_{\text{gp}} = \frac{2}{3 + 2\kappa} = \frac{6 - 2\sqrt{2}}{7}. \quad (9)$$

The end stabilizer is transitive on unoriented edges of each type. Applying (4) separately to the three tree edges and five grandparent edges at o gives

$$\mathbb{E} \deg_{\text{FUSF}(G)}(o) = 3R_{\text{tree}} + 5R_{\text{gp}} = \frac{24 - \sqrt{2}}{7}.$$

Since $(73/50)^2 > 2$, the last expression exceeds $(24 - 73/50)/7 = 161/50$. No mass-transport argument is used.

4 Second calculation: an induced exhaustion and contraction

We independently recover (9) using only induced finite graphs. Put $v_j = p^j(o)$, and let S_n be all descendants of v_n through depth $2n$. Let $G_n = G[S_n]$, $n \geq 2$. When n increases by one, old depths increase by one but the allowed depth increases by two. These finite connected sets are nested and exhaust T , by end-ray coalescence.

Consider descendants of x through depth m , together with its parent y , retaining the direct edge xy but excluding the edge from x to its grandparent. Eliminating the strict descendants gives conductance c_m between x, y , where

$$c_0 = 1, \quad c_m = 1 + \frac{2c_{m-1}}{1 + c_{m-1}}, \quad a_m = 1 + \frac{c_{m-1}}{1 + c_{m-1}} \quad (m \geq 1). \quad (10)$$

Each of the two children supplies its reduced c_{m-1} in series with a unit edge to y , proving the recurrence directly.

Eliminate off-spine vertices in G_n . The resulting conductances on the spine $v_0, \dots, v_n$ are

$$v_0 v_1 : c_n, \quad v_j v_{j+1} : a_{n+j} \quad (1 \leq j < n), \quad v_j v_{j+2} : 1 \quad (0 \leq j \leq n-2).$$

Both children at v_0 are off-spine; above it there is one off-spine child, whose descendant depth is $n + j - 1$. The off-spine part at the top v_n has only one attachment and has no effect; no terminal outside G_n is introduced. Reducing the upper suffix gives

$$u_{n-1,n} = a_{2n-1}, \quad u_{j,n} = a_{n+j} + \frac{u_{j+1,n}}{1 + u_{j+1,n}} \quad (1 \leq j \leq n-2). \quad (11)$$

The final triangle has conductances $c_n, u_{1,n}, 1$.

The map $t \mapsto 1 + 2t/(1+t)$ contracts on $[1, \infty)$ by at most $1/2$, so $c_m \rightarrow c = 1 + \sqrt{2} = A$ and $a_m \rightarrow a = 1 + c/(1+c)$. For the suffix maps in (11) the derivative in their argument is at most $1/4$, all arguments and coefficients are uniformly bounded, and every coefficient index is at least $n+1$. With $\epsilon_n = \sup_{m \geq n+1} |a_m - a|$, iteration gives

$$|u_{1,n} - c| \leq \frac{4}{3} \epsilon_n + C_0 4^{-(n-2)} \rightarrow 0,$$

because $c = a + c/(1+c)$. Apply (7)–(8) with $c_n, u_{1,n}$ and use this induced free exhaustion. The limits are again exactly (9).

For reproducible finite checks, global rational elimination on the actual induced networks gives, at $n = 2$, the triangle $(7/3, 17/10, 1)$ and resistances $27/80, 121/240$. At $n = 3$ it gives $(12/5, 6236/2669, 1)$. These checks support the finite indexing; the written contraction proofs establish the all-depth and infinite-volume assertions.

5 A complete finite certificate for the square

Let W be the radius-10 ball about o in T . Write $K = G[W]$, $J = K^2$. Then $J \subseteq H$. Every H -neighbor of o has a witness using at most two G -edges, hence a tree walk of at most four edges, entirely within the radius-4 tree ball. Therefore J contains *every* H -edge at o .

For clarity, the complete neighbor count is as follows: four strict ancestors; $2 + 4 + 8 + 16 = 30$ strict descendants in depths one through four; the sibling; the uncle (the parent's sibling); the sibling's two children; and the uncle's two children. These disjoint possibilities exhaust two ancestor-or-descendant G steps and total 40. The end stabilizer preserves H , so its degree is 40 everywhere.

Here is an exact specification of the finite data. Index the tree ball by breadth-first lexicographic word order: the root has index 0 and children 1, 2, 3; vertex $i \geq 1$ has outward children $2i + 2, 2i + 3$. Thus

$$|W| = 1 + 3(2^{10} - 1) = 3070, \quad |E(J)| = 19931.$$

Choose the distinguished end ray $0, 1, 4, 10, 22, \dots$. On this ray $p(0) = 1$, $p(i) = 2i + 2$ for $i \neq 0$; off this ray $p(i)$ is the outward-tree predecessor (0 for $i = 1, 2, 3$, otherwise $\lfloor (i - 2)/2 \rfloor$). Construct K by all pairs $i, p(i)$ and $i, p^2(i)$ with endpoints in W , then square K . Order the unordered J -edges (i, j) , $i < j$, lexicographically. The sorted root-neighbor indices are

$$1, \dots, 22, \quad 24, 25, \quad 30, \dots, 45.$$

The source supplement contains the exact compressed JSON certificate named `square_flow_certificate.json.gz`, and a short independent standard-library verifier. For each of those 40 targets v , its row supplies one integer $n_{v,e}$ for every ordered-list edge e . Give that edge its listed orientation $i \rightarrow j$, and put $\theta_v(e) = n_{v,e}/D$, $D = 10^9$. The verifier regenerates the *whole* finite graph from the indexed tree, matches every edge and terminal, and checks the following finite integer identities:

$$\sum_{e \text{ out of } x} n_{v,e} - \sum_{e \text{ into } x} n_{v,e} = D(\mathbf{1}_{\{x=v\}} - \mathbf{1}_{\{x=o\}}), \quad x \in W, \quad (12)$$

$$\sum_v \sum_{e \in E(J)} n_{v,e}^2 = 3208526010913748780. \quad (13)$$

All 797240 current entries are actual integers; all 122800 vertex-divergence equations and all 40 individual energy numerators are verified. Each unoriented edge is counted once. This is feasible-current verification, not trust in a numerical residual or a claim of current optimality. Floating-point solves were used to find the original witnesses; their generator is unnecessary for checking (12)–(13). The immutable data and exact checking suffice.

Lemma 2. *For the currents above,*

$$\mathbb{E} \deg_{\text{FUSF}(H)}(o) \leq \sum_v \mathcal{E}(\theta_v) = \frac{3208526010913748780}{10^{18}} < \frac{321}{100}.$$

Proof. Equation (12) makes θ_v a unit flow from v to o in the finite graph J . Thomson's principle gives $R_J(o, v) \leq \mathcal{E}(\theta_v)$. Take any connected finite induced free exhaustion H_n of H . For large n , $J \subseteq H_n$, so extend θ_v by zero. It remains a unit flow of unchanged energy, giving

$$R_{H_n}(o, v) \leq R_J(o, v) \leq \mathcal{E}(\theta_v).$$

Pass to the free limit and apply (4). Summing over the *complete* root neighborhood gives the claim from (13). The final strict inequality is the integer comparison $100 \cdot 3208526010913748780 < 321 \cdot 10^{18}$. $\square$

Only finitely supported flows were transferred to finite free networks. Minimizing over arbitrary square-summable infinite flows instead would give wired resistance and would not justify the required free upper bound. No boundary is shorted and no wired bound is substituted. Different root edges may use different flows, since only marginals are summed.

Completion of Theorem 1. The graph assertions and (2) were proved above. Lemma 2 proves (3). With (1),

$$b_F(G) - b_F(H) > \frac{1}{2} \left(\frac{161}{50} - \frac{321}{100} \right) = \frac{1}{200}.$$

□

6 Predecessors, scope, and reproducibility

The grandparent graph, finite electrical principles and free-exhaustion framework are inherited, with explicit references above. Tang’s inspected arXiv version proves connectivity and branching properties for this family, not the comparison proved here. Pete–Timár [4, Theorems 1.1–1.2] already show that FUSF *component counts* can depend on the generating set even for a fixed virtually free group; they explicitly distinguish this from expected degree, which is group-invariant for Cayley graphs. Those results are not claimed as new here. Gaboriau’s harmonic-projection invariant is formulated with a transitive unimodular action [3, §2.4]. No such tracial hypothesis is used in our negative calculation. No claim concerning failure in the unimodular subclass is made.

The original question was checked in the actual seven-page 2010 source, including its unrestricted scope and Cayley comment. Our bounded primary-source investigation through 22 October 2026 authenticated no earlier publication of this complete expected-degree square comparison. That does not certify historical firstness or continued worldwide openness. Tang’s final journal PDF and Pete–Timár’s final journal PDF returned access-challenge pages; their actual arXiv versions identified below were read instead. The full Lyons–Peres book and all unindexed sources were not inspected. Source-edition and exact read-scope limits are recorded in the supplement. None of these dependency DOIs is presented as a prior publication of this full counterexample.

The supplement contains the entire rational-current data, the independent indexed-tree verifier, exact per-target outcomes, an independent binary-word/global-rational-elimination check of the finite reductions, editable manuscript, dependencies, priority notes, licenses and offline packaging instructions. Every current certificate entry is checked exactly; no external solver, numerical tolerance, original claimant program or installed third-party package is needed for replay. The certificate proves a finite statement; Lemma 2 is the separate universal argument making it an infinite free-forest bound. The finite circuit diagnostics do not prove the infinite limits; Sections 3–4 do. No formal proof-assistant certification is claimed.

Disclosure and license. AI tools were extensively involved in mathematical research and exploration, proof development and reconstruction, computational verification, literature and source investigation, manuscript preparation, editing, and adversarial auditing. Alec Kriebel is the named human author. This is an unrefereed preprint and has not undergone formal independent human peer review. AI reviews do not supply human refereeing or journal acceptance. Original prose and certificate data are CC BY 4.0; original verification and packaging code is MIT licensed.

References

- [1] M. Abért, *Some questions*, 2 November 2010, printed p. 4, Question 24. Author-hosted problem list.
- [2] I. Benjamini, R. Lyons, Y. Peres, and O. Schramm, *Uniform spanning forests*, Ann. Probab. **29** (2001), 1–65. doi:10.1214/aop/1008956321. Inspected corrected author version dated 27 January 2009: usf.pdf.
- [3] D. Gaboriau, *Invariant percolation and harmonic Dirichlet functions*, Geom. Funct. Anal. **15** (2005), 1004–1051. doi:10.1007/s00039-005-0539-2. Inspected arXiv:math/0405458v3.
- [4] G. Pete and Á. Timár, *The free uniform spanning forest is disconnected in some virtually free groups, depending on the generator set*, Ann. Probab. **50** (2022), 2218–2243. doi:10.1214/22-AOP1581. Actual inspected body: arXiv:2006.06387v2, 24 January 2021.
- [5] P. Tang, *Weights of uniform spanning forests on nonunimodular transitive graphs*, Electron. J. Probab. **26** (2021), 1–62. doi:10.1214/21-EJP709. Actual inspected body: arXiv:1908.09889v2, 3 December 2020.