

A continuous free product system not generated by its tensor core

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Abstract

We construct an associative pointed free product system of separable complex Hilbert spaces whose maximal ordered tensor subsystem does not generate the whole free system. Its tensor core is the time-ordered finite-word Fock system, and the closed free subsystem generated by that core is precisely the full finite-word Fock system. The ambient system has, in addition, nonzero orthogonal sectors of words of ordinal length below ω^2 . A recoverable Gaussian mixture supplies a sigma-finite measure class invariant under finite-prefix absorption. Finite monochromatic-run charts turn concatenation into the free product structure. We prove separability, an explicit continuous Hilbert-field structure including time zero, and continuity of the full structural maps and their inverses. This gives a negative answer, in the scalar specialization, to the universal tensor-core generation question stated by Skeide in 2009. It does not answer the separate adapted-cocycle or forward tensor-recovery questions. The argument is analytic; accompanying finite computations do not certify its infinite-dimensional assertions. This draft is unrefereed and extensively AI-assisted.

1 The question, conventions, and result

For pointed complex Hilbert spaces (G_i, ω_i) , write $G_i^\circ = G_i \ominus \mathbb{C}\omega_i$. Their reduced pointed free product is

$$G_1 \otimes G_2 = \mathbb{C}\omega \oplus \bigoplus_{k \geq 1} \bigoplus_{\substack{\epsilon \in \{1,2\}^k \\ \epsilon_j \neq \epsilon_{j+1}}} G_{\epsilon_1}^\circ \otimes \cdots \otimes G_{\epsilon_k}^\circ. \quad (1)$$

We use the usual associator obtained by flattening centered alternating words. The ordered tensor inclusion $G_1 \otimes G_2 \hookrightarrow G_1 \otimes G_2$ has range $\mathbb{C}\omega \oplus G_1^\circ \oplus G_2^\circ \oplus (G_1^\circ \otimes G_2^\circ)$.

A pointed free product system is a family $(E_t, \omega_t)_{t \geq 0}$ with pointed unitaries

$$u_{s,t} : E_s \otimes E_t \longrightarrow E_{s+t}, \quad E_0 = \mathbb{C}\omega_0,$$

satisfying associativity and the canonical zero-time identifications. This is the scalar case of Skeide's Hilbert-bimodule definition [1, pp. 529–530]; scalar vectors are central and the maps are coefficient-bilinear. Our convention puts the s -factor in the later interval $(t, s+t)$ and the t -factor in $(0, t)$.

For a finite ordered time partition π of t , let $T_\pi \subset E_t$ be the range of the iterated ordered tensor inclusion. Its tensor core is the closed subspace intersection $\bigcap_\pi T_\pi$. We compute this intersection directly and prove it is a tensor subsystem; no general assertion about intersections of Hilbert-module tensor products is needed. By the free subsystem generated by a family we mean the smallest closed free subsystem containing it, equivalently here the closed span of its finite-partition free-product images.

Theorem 1. *There is a pointed free product system of separable complex Hilbert spaces with the following properties for every $t > 0$:*

(i) *Its maximal ordered tensor subsystem is*

$$F_t = \mathbb{C}\omega \oplus \bigoplus_{m \geq 1} L^2(\Delta_m(t), dx), \quad \Delta_m(t) = \{t > x_1 > \cdots > x_m > 0\}.$$

(ii) *The closed free subsystem generated by F is*

$$K_t = \mathbb{C}\omega \oplus \bigoplus_{m \geq 1} L^2((0, t)^m, dx).$$

(iii) $K_t \subsetneq E_t$, *with a nonzero orthogonal grade- ω sector.*

The system has the explicit continuous-field convention proved in Section 5: norm-continuous support-subspace sections, a continuous vacuum, local triviality at positive time, and continuous full structural maps and inverses. The same field structure restricts to F and K .

Skeide sketched the free-product-system viewpoint earlier in the white-noise discussion of [2, Section 10.2]. The 2009 contribution explicitly asks whether every free system is generated by its tensor core and gives the affirmative Fock example [1, p. 530]. Its subsequent construction from a supplied spatial tensor system and its recovery question run in the opposite direction. Later references to that forward construction and the historical limits of this draft are discussed in Section 8.

2 A recoverable sigma-finite sequence measure

Let λ denote Lebesgue measure on $\mathbb{R}$, let φ be the standard normal density, and let γ be the product standard normal probability measure on $\mathbb{R}^{\mathbb{N}}$, where $\mathbb{N} = \{0, 1, \dots\}$. Put

$$\Gamma = \left\{ z : 2^{-j} z_j \rightarrow 0, \quad \frac{1}{N} \sum_{j < N} z_j^2 \rightarrow 1 \right\}.$$

This is Borel and $\gamma(\Gamma) = 1$: Gaussian tail bounds and Borel–Cantelli prove the first condition, and the strong law proves the second. These are standard probability inputs; see, for example, [9]. On $\mathbb{R} \times (0, \infty) \times \Gamma$ use $da \, d\delta / \delta \gamma(dz)$ and define

$$\Theta(a, \delta, z)_j = a + \delta 2^{-j} z_j.$$

Let $C \subset \mathbb{R}^{\mathbb{N}}$ consist of the sequences for which

$$a(x) = \lim_j x_j \in \mathbb{R}, \quad d(x)^2 = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{j < N} 4^j (x_j - a(x))^2 \in (0, \infty). \quad (2)$$

All limits and their existence conditions are Borel. The map Θ is a Borel bijection onto C , with inverse

$$a = a(x), \quad \delta = d(x), \quad z_j = 2^j (x_j - a(x)) / d(x).$$

Indeed, these inverse coordinates satisfy exactly the two conditions defining Γ . Let $\mu = \Theta_*(da \, d\delta / \delta \gamma)$. The sets $|a| \leq N$, $N^{-1} \leq d \leq N$ exhaust C and have measure $4N \log N < \infty$ for $N \geq 2$. Thus μ is sigma-finite on a standard Borel space. In particular, this is a mixture over recoverable parameters; equivalence of distinct fixed-parameter infinite Gaussian laws is never used.

Lemma 2 (Finite-prefix absorption). *For each $k \geq 0$, prepending k real coordinates is a Borel bijection $p_k : \mathbb{R}^k \times C \rightarrow C$. For every nonnegative Borel f ,*

$$\int_C f \, d\mu = \int_C \int_{\mathbb{R}^k} f(p_k(v, y)) q_k(v; y) \, dv \, \mu(dy), \quad (3)$$

where $q_0 = 1$ and

$$q_k(v; y) = \prod_{j=0}^{k-1} \frac{\varphi((v_j - a(y))/(2^{k-j}d(y)))}{2^{k-j}d(y)}. \quad (4)$$

These densities are finite and strictly positive, so the product and output measures are equivalent in both directions. Moreover,

$$q_{k+\ell}(v, w; y) = q_k(v; p_\ell(w, y)) q_\ell(w; y). \quad (5)$$

Proof. Deleting k coordinates preserves a and changes d to $2^{-k}d$. In the second limit in (2), this is the factor 4^{-k} , after deletion of finitely many summands and a harmless ratio of Cesaro lengths. Conversely, a finite prefix preserves the limit and multiplies the tail scale by 2^k . These inverse operations are Borel.

In parameter coordinates, the tail has parameters $(a, 2^{-k}\delta, (z_k, z_{k+1}, \dots))$. The measure $d\delta/\delta$ is invariant under this scale change, and the Gaussian tail again has law γ . Split the first k independent Gaussian coordinates from the tail and change variables in these k coordinates. Since the original scale is $2^k d(y)$, the coordinate scales are $2^{k-j}d(y)$ and their densities are (4). Tonelli gives (3), without requiring a probability disintegration of a measure of infinite total mass. Each density integrates to one in v and is everywhere positive and finite. Finally $d(p_\ell(w, y)) = 2^\ell d(y)$ makes the first k scales in (5) exactly the first k scales of $q_{k+\ell}$; the remaining ℓ factors agree as well. $\square$

For any Borel bijection $c : (X, \eta) \rightarrow (Y, \zeta)$ with two-way measure-class equivalence, its canonical unitary is

$$U_c f = (f \circ c^{-1}) \sqrt{\frac{d(c_*\eta)}{d\zeta}}. \quad (6)$$

The Radon–Nikodym chain rule and the positive square root give $U_{boc} = U_b U_c$. For the prefix map in Lemma 2, (3) shows that the multiplier from product measure to μ is $q_k^{-1/2}$, rather than $q_k^{1/2}$. Although this multiplier need not be bounded pointwise, the change-of-measure identity makes the map unitary. This standard L^2 transport principle is the only density mechanism used below.

Lemma 3. *Common translation preserves μ , and common dilation by $r > 0$ satisfies $\mu(rA) = r\mu(A)$. Their canonical unitaries are strongly continuous on $L^2(C, \mu)$. For each fixed $c \in \mathbb{R}$, almost every sequence has no coordinate equal to c , has limit different from c , and changes sides of c only finitely many times.*

Proof. Translation changes a to $a + h$. Dilation changes (a, δ) to $(ra, r\delta)$, contributing the factor r from da . In coordinates $(a, \log \delta, z)$, the unitaries are the usual finite-dimensional translation and dilation operators tensored with the identity on the Gaussian factor. Strong continuity follows first on compactly supported continuous functions of $(a, \log \delta)$ times an $L^2(\gamma)$ vector, then by density. The events $a = c$ and $x_j = c$ are null by Lebesgue measure and the finite Gaussian density, respectively. Off their countable union, convergence to $a \neq c$ makes the side eventually constant. $\square$

Lemma 4 (Rises in an infinite block). *For μ -almost every x , both $x_{2j+1} > x_{2j}$ and $x_{2j+1} < x_{2j}$ occur infinitely often.*

Proof. The sign of $x_{2j+1} - x_{2j}$ is that of $z_{2j+1}/2 - z_{2j}$. These variables, for distinct j , are independent symmetric nondegenerate Gaussians. Each sign has probability $1/2$, so Borel–Cantelli gives the claim. It holds for every a, δ on the same Gaussian probability-one set, and Tonelli integrates it against the sigma-finite parameter measure. Restricting to any measurable interval event preserves this almost-everywhere assertion. $\square$

3 Ordinal word spaces and fixed-cut factorization

Every $\alpha < \omega^2$ uniquely equals $\omega n + m$ for finite $n, m \geq 0$. Represent a word of that grade by n consecutive infinite sequences and a final string of m letters. Define

$$\Omega_\alpha = C^n \times \mathbb{R}^m, \quad \nu_\alpha = \mu^n \times \lambda^m.$$

For $\alpha = 0$, use the singleton empty word of mass one. Let (Ω, ν) be their countable measured disjoint union and $H = L^2(\Omega, \nu)$. These standard Borel sigma-finite sectors have separable L^2 spaces: a countable generating algebra and a finite-measure exhaustion provide a countable dense set of simple functions. Hence H is separable. Its vacuum ω is the indicator of the empty word. For an open interval I , let $\Omega_\alpha(I)$ require every letter coordinate to lie in I , and put $E(I) = L^2(\Omega(I), \nu) \subset H$. The empty word is always included.

Lemma 5 (Coordinate-retaining concatenation). *For a fixed finite list $\alpha_1, \dots, \alpha_r < \omega^2$, concatenation is a Borel bijection*

$$c_{\alpha_1, \dots, \alpha_r} : \prod_i \Omega_{\alpha_i} \longrightarrow \Omega_{\alpha_1 + \dots + \alpha_r}$$

with two-way measure-class equivalence. The canonical unitaries for these fixed lists compose associatively under regrouping.

Proof. For two grades, ordinal addition gives

$$(\omega n + m) + (\omega p + q) = \begin{cases} \omega(n + p) + q, & p > 0, \\ \omega n + (m + q), & p = 0. \end{cases}$$

If $p > 0$, write the two inputs as $(x^1, \dots, x^n; v)$ and $(y^1, \dots, y^p; w)$. The output is

$$(x^1, \dots, x^n, p_m(v, y^1), y^2, \dots, y^p; w).$$

All coordinates are retained. The inverse removes the specified first m coordinates of the specified $(n + 1)$ st block. Lemma 2 proves bijectivity and measure-class equivalence. If $p = 0$, only finite suffix concatenation occurs. Iteration proves the general statement.

Regrouping retains the same original infinite blocks; every finite string immediately before one becomes its prefix, and the final finite string remains the suffix. Thus the coordinate maps agree for any parenthesization. The chain rule in (6), explicitly (5) for a repeated prefix absorption, proves equality of the unitaries. This does not assert cancellation for ordinal addition or a common inverse when the input grades are allowed to vary. $\square$

Lemma 6 (Run charts). *For $s, t > 0$, concatenation induces a pointed unitary*

$$E((t, s + t)) \otimes E((0, t)) \longrightarrow E((0, s + t)).$$

Proof. In each output sector remove all coordinate and block-limit equalities to the fixed cut t . The removed set is null by Lemma 3 and sigma-finiteness of the other factors. Color each remaining coordinate by its side of t . Every infinite block is eventually one color and has

finitely many changes. There are finitely many blocks and a finite suffix, so the entire ordinal word has finitely many maximal nonempty monochromatic convex runs. There can be a color change at a limit position ωi , even without a predecessor there; it contributes at most one additional boundary.

Each run is again an allowed word. If it starts at $\omega i + k$ and reaches a later block, it consists of the tail from position k in its first block, any full intervening blocks, and a possible finite prefix in its last block or finite suffix. The first tail lies in C by Lemma 2; all full blocks remain separate. Its grade is below ω^2 . A run contained in one block before its limit is finite. This accounts also for runs crossing several limit boundaries.

Conversely, concatenating any finite alternating list of nonempty one-color words yields exactly those maximal runs, after the same null exclusions. Run count, color pattern, and ordinal run lengths give countably many charts. For each chart the domain is a Borel product of interval-support sets, and its image is Borel by the fixed-grade Borel isomorphism in Lemma 5. Its restrictions preserve measure classes in both directions. The images of distinct charts are disjoint because maximal runs are unique, even when their lists have the same ordinal sum. They cover the output up to a null set. Taking the canonical chart unitaries and their orthogonal direct sum is exactly (1), and is onto. $\square$

4 The associative free product system

Set $E_t = E((0, t))$ for $t > 0$, $E_0 = \mathbb{C}\omega$, and $\omega_t = \omega$. Translate every coordinate in E_s by t and apply Lemma 6 to define $u_{s,t}$. At zero use the canonical identifications. Translation preserves μ and finite-letter Lebesgue measure, so these are pointed unitaries.

Proposition 7. *The maps $u_{s,t}$ satisfy the full free-product associativity identity.*

Proof. For three positive lengths r, s, t , the original factors occupy the intervals $(s + t, r + s + t)$, $(t, s + t)$ and $(0, t)$. Delete the null sets for the two fixed cuts. Every word then has a finite maximal-run decomposition in the three interval colors. The canonical free-product associator identifies both parenthesizations with the same direct sum of finite words of the three centered spaces having unequal neighboring colors. Grouping two factors groups maximal strings avoiding the third; their internal multiplication is the same concatenation. Lemma 5 and the positive Radon–Nikodym chain rule therefore identify the two operator compositions on each countable grade chart and hence on the whole space.

The translations also agree: the three original factors are shifted by $s + t$, t , and 0 in either composition. Common translation commutes with concatenation, and $q_k(v + h; y + h) = q_k(v; y)$. Thus no extra density factor arises. Zero parameters and vacua reduce to the canonical identities. The same reasoning works for every finite regrouping. Only finitely many fixed cuts are used for any one identity, so a conull set uniform over all real cuts is neither asserted nor required. $\square$

5 An explicit continuous-field structure

Let P_t be the projection on H given by the indicator of all coordinates belonging to $(0, t)$ on every nonempty sector and by the identity on the vacuum; let P_0 be the vacuum projection P_{vac} . We use the field $E_t = P_t H$ whose sections are the norm-continuous H -valued functions $x(t) \in P_t H$. This specifies the continuity convention, rather than leaving an additional bundle axiom implicit.

Proposition 8. *The projections P_t are strongly continuous for $t \geq 0$. They define a separable continuous Hilbert field, locally trivial for $t > 0$, with continuous vacuum section.*

Proof. The coordinates of a word are finitely many convergent sequences and a finite suffix. Their closure is compact; its supremum is a coordinate or one of the finitely many block limits. For each fixed positive t , the event that the supremum equals t is therefore null, by a countable union of coordinate equalities and finitely many limit equalities. Hence the interval indicators converge almost everywhere when $t_j \rightarrow t > 0$. At zero, every nonempty word with all coordinates positive has a positive coordinate and is eventually excluded as $t_j \rightarrow 0$; a word with a nonpositive coordinate is always excluded. Dominated convergence against each $|h|^2$ proves strong continuity, without assuming finite total measure of a sector.

The sections $P_t h$ are continuous and pass through every fibre vector. A countable dense set of $h \in H$ gives a countable fundamental family. These sections give exactly the stated ambient norm-continuous field by local approximation at a fibre value.

On grade $\omega n + m$, dilation satisfies $\nu_\alpha(rA) = r^{n+m}\nu_\alpha(A)$, since each infinite block contributes one factor from da and each finite letter one from Lebesgue measure. Its canonical unitary is

$$D_r f(x) = r^{-(n+m)/2} f(x/r).$$

Lemma 3 and ordinary finite-dimensional dilation give strong continuity on each sector. Finite-sector approximation extends it to H . The maps $D_t : E_1 \rightarrow E_t$ trivialize the field at positive time. No local triviality at the dimension-one fibre at zero is asserted. $\square$

Proposition 9. *The full free-product structural maps and their inverses are continuous between the source and target fields, jointly in $(s, t) \in [0, \infty)^2$. In particular products of continuous ordered tensor sections and elementary free-word sections are continuous.*

Proof. Put $J = H \otimes H$ and $A_t = P_t - P_{\text{vac}}$ on H° . On the vacuum of J put $Q(s, t) = 1$; on each alternating-word summand put $Q(s, t)$ equal to the tensor product of the appropriate copies of A_s and A_t . Each such tensor product is strongly continuous. Approximation by finitely many free-word summands proves strong continuity of Q , whose range is the embedded $E_s \otimes E_t$. Extend the structure map by zero off that range:

$$V(s, t) = u_{s,t} Q(s, t) : J \longrightarrow H.$$

The run-chart construction already proves that these are partial isometries with $\|V(s, t)\| \leq 1$, initial projection $Q(s, t)$, and final projection P_{s+t} .

To prove strong continuity, take first a vector in a fixed alternating pattern and fixed ordinal input grades, an elementary tensor of fixed ambient vectors. After $Q(s, t)$, translate the coordinates from the first copy by t and leave the others fixed. Apply the fixed-list concatenation unitary of Lemma 5. Strong continuity of P and translation makes the result jointly norm-continuous. Finite sums of such vectors are dense in J . The uniform norm bound on the already defined $V(s, t)$ passes continuity to all of J . Equivalently, finite pattern-and-grade truncations commute with Q . We do not assume a bounded unrestricted concatenation map across all patterns with potentially coincident output grades.

This argument includes zero parameters. On a fixed elementary term containing a centered factor from a fibre tending to zero, $A_t h \rightarrow 0$, so that term vanishes. A term from the surviving single factor has its usual translated limit; the vacuum stays fixed. The norm bound handles the full sum.

Strong continuity of V alone would not imply that of its adjoint. Here its continuous final projections give, for $p = (s, t) \rightarrow p_0 = (s_0, t_0)$ and fixed $h \in H$,

$$\begin{aligned} \|V(p)^* h - V(p_0)^* h\|^2 &= \|P_{s+t} h\|^2 + \|P_{s_0+t_0} h\|^2 \\ &\quad - 2 \operatorname{Re} \langle h, V(p) V(p_0)^* h \rangle \longrightarrow 0. \end{aligned}$$

Thus V^* is strongly continuous as well. Restrictions of V and V^* to the continuous source and target fields prove the assertion. For variable continuous vectors, the usual norm estimate adds

their norm difference to the fixed-vector estimate. The ordered tensor inclusion is a continuous restriction of these maps. $\square$

6 Computing the maximal tensor core

For a finite partition π of $(0, t)$, order its cells from latest to earliest. Its ordered tensor inclusion allows at most one centered word from each cell, in that order. The run-chart unitary therefore identifies T_π with the entire support subspace of words whose interval colors are nonincreasing in real time along their ordinal positions. Positive finite density multipliers preserve these support subspaces.

Proposition 10. *For every $t > 0$, $\bigcap_\pi T_\pi = F_t$ as in Theorem 1.*

Proof. First intersect just the two-cell partitions at rational $c \in (0, t)$. Their countably many boundary exceptions have a null union. On its complement, a word belongs to every such tensor support precisely when its real labels are nonincreasing throughout its ordinal order. Indeed, if an earlier coordinate is smaller than a later coordinate, a rational strictly between the two values detects a low color followed by a high color. Conversely nonincreasing labels pass every cut. The index set of any one word is countable. Intersections of these countably many multiplication projections give exactly the L^2 support on that nonincreasing-label set.

Any sector with at least one infinite block has measure zero on this set by Lemma 4; sigma-finiteness justifies the product null-set statement. Restricting to interval events cannot revive a null set. In finite grade m , equal-coordinate diagonals are Lebesgue-null, so the remaining support is $\Delta_m(t)$. Thus the rational-cut intersection is F_t , and the intersection over all partitions is contained in F_t .

For the reverse inclusion, fix any one finite partition, with arbitrary real endpoints. Almost every decreasing finite tuple avoids its boundaries and has the required cell order. Its full simplex L^2 subspace is therefore in T_π . Hence $F_t \subset T_\pi$ for each individual π . This is an inclusion of closed subspaces and takes no uncountable union of exceptional null sets. It proves the desired equality. $\square$

Proposition 11. *With inherited multiplication, F is a continuous tensor product subsystem and is maximal among such tensor subsystems.*

Proof. Every strictly decreasing finite word in $(0, s + t)$ splits uniquely at t into its later labels followed by its earlier labels, except on the null boundary. Splitting each simplex according to the number of later labels identifies its L^2 space with the orthogonal sum of the corresponding tensor products of simplexes. All measures in these finite grades are ordinary Lebesgue measures, so concatenation has density one. Summing over total word length proves that $F_s \otimes F_t \rightarrow F_{s+t}$ is onto and isometric. Associativity and continuity follow from the ambient maps.

More explicitly, the fixed global projection on finite decreasing words commutes with every P_t and with positive dilation. Its fibre restrictions therefore form the continuous subfield of F . Any tensor subsystem with the inherited maps has its whole time- t fibre in each iterated partition tensor range T_π , and hence in their intersection F_t . This proves maximality. $\square$

The ordered-simplex realization is the usual time-ordered finite-particle Fock system. Its tensor-system role is classical; see [10, 2]. The full finite-word space in the next section is the full Fock space over $L^2((0, t))$. The extra ordinal sectors, rather than either classical Fock identification, provide the obstruction to generation.

7 The generated free hull and the orthogonal obstruction

Let K_t be the finite-word space in Theorem 1. Its fixed global grade projection commutes with P_t and dilation, so K is a continuous closed subfield. A finite word has finitely many finite color runs, and a finite list of finite runs concatenates to a finite word. Both directions of the fixed-cut unitary therefore restrict to $K_s \otimes K_t \simeq K_{s+t}$. Consequently K is a closed free subsystem and contains F .

Proposition 12. *The closed free subsystem generated by F is exactly K , and $K_t \neq E_t$ for every $t > 0$.*

Proof. Every finite-partition free word from F has finite total word length, so its image lies in K . Conversely, fix a finite length m . Almost every tuple in $(0, t)^m$ has pairwise distinct coordinates. A partition with rational interior endpoints can separate their values into distinct cells. For each assignment of these m cells to the m positions, the full support rectangle is the image of the alternating singleton sectors of the corresponding F -fibres, under the finite-partition free-product inclusion. Different positions have different colors, so the word is reduced. Translation gives the singleton space of each cell. Thus the whole L^2 subspace on that rectangle belongs to the range, not merely its individual coordinate tuples.

There are countably many rational partitions and cell assignments. Their supports cover all distinct-coordinate tuples. The closed span of their support ranges is therefore $L^2((0, t)^m)$: a vector orthogonal to all ranges vanishes on their countable union. Summing over m proves the hull is K . Since K is already closed and orthogonal to every infinite grade, completion cannot add an infinite-grade vector.

It remains to exhibit a nonzero L^2 vector in grade ω supported in $(0, t)$. Restrict the parameter region to

$$a \in [t/3, 2t/3], \quad \delta \in [t/100, t/50].$$

It has measure $(t/3) \log 2$. Require also $|z_j| < (t/(3\delta))2^j$ for every j . The thresholds are at least $(50/3)2^j$. Chebyshev's inequality and the countable union bound give a uniform failure probability at most

$$\sum_{j \geq 0} \frac{9}{2500 \cdot 4^j} = \frac{3}{625}.$$

Thus the image $S_t \subset C$ of this Borel region satisfies

$$0 < \frac{622}{625} \frac{t}{3} \log 2 \leq \mu(S_t) \leq \frac{t}{3} \log 2 < \infty.$$

Intersecting with Γ changes no probability. The strict inequalities on the Gaussian coordinates imply $0 < a + \delta 2^{-j} z_j < t$, so every resulting word lies in $(0, t)$. The indicator 1_{S_t} is a nonzero square-integrable grade- ω vector, orthogonal to K_t . This proves properness and completes Theorem 1. $\square$

8 Historical scope and verification limits

The negative answer here concerns the universal generation assertion in [1, p. 530]. It is already obtained over $\mathbb{C}$, so it also precludes an affirmative universal statement over all coefficient algebras. We make no claim about adapted cocycles, characterization of the associated noises, or recovery of a supplied spatial tensor system after the forward tensor-to-free construction. We prove the explicit regularity convention in Section 5; no arbitrary additional regularity axiom is implicitly included.

There is a simple reason why the known forward construction itself cannot provide this obstruction, even from a non-Fock tensor input. Write $G(T) = \text{Hull}(T)$ for the free system

generated by a supplied embedded tensor system T , and let C be its tensor intersection. For every partition the tensor multiplication of T identifies T_t with an embedded ordered product of its smaller fibres; thus $T_t \subset C_t$. Monotonicity of the closed free hull gives

$$G(T) = \text{Hull}(T) \subset \text{Hull}(C) \subset G(T).$$

Consequently the maximal tensor core of $G(T)$ generates all of $G(T)$. This excludes that particular direct transfer from old tensor examples; it does not rule out every construction from the earlier literature.

Skeide's later literature cites a work entitled *Free product systems generated by spatial tensor product systems* as in preparation; see the bibliography of [3] and [4, Section 4 and reference 30]. A 2 February 2026 seminar abstract [5] describes the noise-from-free-system and spatial-tensor-to-generated-free-system constructions. Its full abstract contains neither the universal generation converse nor the present negative example. The exact in-preparation manuscript was not recovered, and no recording, slides or transcript of the talk were inspected. These limits prevent an inference about unseen manuscript or talk content. In the bounded primary-source comparison accompanying this paper, no earlier instance of the particular counterexample was authenticated. This is not a certificate of global firstness or a claim about worldwide openness in 2026; unpublished, unindexed or differently formulated work remains outside that conclusion.

Classical full-Fock rectangle approximation, for example [6, Section 3 and Theorem 1(1)], concerns a specified ambient full Fock space consisting of finite Cartesian-power sectors. It does not imply density in the extra ordinal sectors of our E . Nonclassical tensor noises and stationary factorizing measure types are substantial prior constructions [7, 8]. Their tensor factorization maps and measure-class half-density machinery should be distinguished from the pointed free-product run maps required here. A non-Fock tensor remainder alone does not supply these free maps or the proper free hull of the maximal tensor intersection.

Terminology alone can mislead. Fowler's *Free E_0 -semigroups* [11] studies the associated Arveson tensor product system; it is not the arbitrary pointed free-product-system category used here. Arici and Ge's *Quadratic subproduct systems, free products, and their C^* -algebras* [12] uses a free product operation for discrete quadratic subproduct systems, a different object and question. Neither terminological comparison by itself establishes priority. The Gaussian probability tools, sigma-finite change of measure, positive half-density transport, and classical Fock constructions are standard inputs rather than claims of new techniques.

The source comparison record states actual reading scopes rather than equating keyword screens with complete paper readings. It records 60 overlapping executed query inputs and 26 retrieval requests, of which 24 returned bodies, including navigation pages and an unrelated thesis. These are search-custody counts, not counts of independent works or whole papers read. The accessible exact passages support the comparisons just stated.

The theorem is proved by the analytic arguments above. The historical source packet includes a program checking 4,240 bounded ordinal and Gaussian-scale identities. Its assertions are removed by optimized Python, so an optimized run of that historical script is not verification. The accompanying explicitly guarded controls test finite support combinatorics and density identities; neither they nor the old program certify infinite Gaussian laws, measure-class equivalence, strong continuity, or the full theorem. Reproduction documentation must preserve this distinction.

Authorship, AI assistance, and review status. Alec Kriebel is the named human author and independent researcher. AI systems were used extensively in research planning, proof development and reconstruction, literature searching and comparison, computational checks, manuscript writing, and adversarial auditing. These roles were substantive, rather than limited to copyediting. The accompanying record distinguishes inherited-context AI audits from any

later review without that history; automated agreement is not independent human peer review. This draft is unrefereed, has received no formal independent human peer review, and carries no claim of formal proof-assistant certification. Mathematical and historical claims remain open to correction.

References

- [1] M. Skeide, Free Product Systems, in *Mini-Workshop: Product Systems and Independence in Quantum Dynamics*, Oberwolfach Reports **6** (2009), 528–530.
[doi:10.4171/OWR/2009/09](https://doi.org/10.4171/OWR/2009/09).
- [2] M. Skeide, *The index of (white) noises and their product systems*, preprint (2006), especially Section 10.2. [arXiv:math/0601228](https://arxiv.org/abs/math/0601228).
- [3] M. Skeide, *Classification of E_0 -semigroups by product systems*, preprint (2009), revision of 13 May 2014, bibliography entry [Ske14b]. [arXiv:0901.1798v4](https://arxiv.org/abs/0901.1798v4).
- [4] M. Skeide, Spatial Markov semigroups admit Hudson–Parthasarathy dilations, *SIGMA* **18** (2022), 071, 8 pp. [doi:10.3842/SIGMA.2022.071](https://doi.org/10.3842/SIGMA.2022.071).
- [5] M. Skeide, *Noises, Spatial Product Systems, and Free Product Systems*, seminar abstract, Saarland University, 2 February 2026. [Institutional abstract](#).
- [6] S. Attal and I. Nechita, *Discrete approximation of the free Fock space*, preprint (2008). [arXiv:0810.4070](https://arxiv.org/abs/0810.4070).
- [7] B. Tsirelson, *Nonclassical stochastic flows and continuous products*, preprint (2004), revision of 18 December 2004. [arXiv:math/0402431](https://arxiv.org/abs/math/0402431).
- [8] V. Liebscher, *Random sets and invariants for (type II) continuous tensor product systems of Hilbert spaces*, preprint (2003). [arXiv:math/0306365](https://arxiv.org/abs/math/0306365).
- [9] O. Kallenberg, *Foundations of Modern Probability*, third edition, Springer, 2021.
[doi:10.1007/978-3-030-61871-1](https://doi.org/10.1007/978-3-030-61871-1).
- [10] W. Arveson, *Noncommutative Dynamics and E -Semigroups*, Springer Monographs in Mathematics, Springer, 2003. [doi:10.1007/978-0-387-21524-2](https://doi.org/10.1007/978-0-387-21524-2).
- [11] N. J. Fowler, Free E_0 -semigroups, *Canadian Journal of Mathematics* **47** (1995), 744–785.
[doi:10.4153/CJM-1995-039-0](https://doi.org/10.4153/CJM-1995-039-0).
- [12] F. Arici and Y. Ge, Quadratic subproduct systems, free products, and their C^* -algebras, *Integral Equations and Operator Theory* **98** (2026), article 19.
[doi:10.1007/s00020-025-02821-x](https://doi.org/10.1007/s00020-025-02821-x).