A free-chain counterexample to a Nash-intersection conjecture

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22 October 2026
Version 1.0.0 — unrefereed preprint

Abstract

We compute the intersection of the closed Nash sets of the complete family of prime divisors obtained by n successive free point blowups over $(\mathbb{C}^2, 0)$, after fixing a prefix of k centers, where $2 \le k < n$. In suitable coordinates the intersection is $\{\operatorname{ord}_t x \ge \lceil n/k \rceil,\ \operatorname{ord}_t y \ge n\}$. For five blowups, fixing two or three centers gives the thresholds $(3, 5)$ or $(2, 5)$. Neither is $q(1, j)$ for positive integers q, j. These examples disprove the specified-reference-multiple assertion in Conjecture 1 of Fernández de Bobadilla, Pe Pereira and Popescu-Pampu, under each contact and indexing convention explicitly treated here. The source's bare equivalence symbol is undefined and its indexing is inconsistent; no unique author-intended correction is asserted. The intersections are themselves maximal divisorial sets for other toric divisors. Thus the result does not disprove an assertion allowing an arbitrary divisor, or solve the generalized Nash problem.

1 The question and the precise negative answer

For a prime exceptional divisor E over $(\mathbb{C}^2, 0)$, let $\overline{N}_E$ be the Zariski closure of the nonconstant arcs whose lifts to its minimal point-blowup model meet E. We work throughout with closed sets of arcs, equivalently reduced supports, in the centered arc space

$$\mathcal{A} = \{(x(t), y(t)) : x, y \in t\mathbb{C}[[t]]\}, \qquad \operatorname{ord}_t 0 = +\infty.$$

All cylinders below are defined by finitely many coefficient equations. For $q \ge 1$, write $\overline{\operatorname{Cont}}^q(E)$ for the closure of arcs whose lifts meet E with exact intersection multiplicity q. We will also check the convention using multiplicity at least q.

Conjecture 1 in [1, p. 31], and in the workshop account [2, p. 2679], proposes that an intersection of Nash sets of divisors of the same combinatorial type with a prescribed common prefix equals $\overline{\operatorname{Cont}}^q(E_p)$, where p is a chosen free point and E_p is the divisor created by blowing it up. The surrounding definition in [1, p. 30] defines $E' \equiv_{\ge i} E$ by equal combinatorial type and at least i common blown-up centers. The conjecture itself prints a bare $\equiv_{i_0}$, without defining that symbol there. Remark 4.14 indexes the first-exceptional case in a way that does not agree with a mechanical zero-based indexing of all created centers. These are real limits on interpretation.

To avoid building a conclusion on an invented correction, we index centers $c_1 = 0, c_2, \ldots, c_n$, and let D_j be created at c_j. Continue the zero reference chain if D_j with $j > n$ is mentioned. A prefix of length k includes c_k. The examples below treat both fixing or not fixing the selected free c_3, both common-prefix and conditional exact-contact families, and every reference D_j. They are counterexamples to all the expressly listed readings, not an assertion about an unspecified

replacement of the printed conjecture. The repository catalogue's union symbol is a transcription discrepancy: both inspected primary formulations use an intersection.

Theorem 1. *Let $D_1, \dots, D_n$ be the free chain following $y = 0$, with $n \geq 3$, and let $2 \leq k < n$. Let $\mathcal{F}_{n,k}$ be the complete family of final prime divisors with this all-free combinatorial type and with the first k centers fixed. Then*

$$I_{n,k} := \bigcap_{E' \in \mathcal{F}_{n,k}} \overline{N}_{E'} = \{(x, y) \in \mathcal{A} : \operatorname{ord}_t x \geq \lceil n/k \rceil,\ \operatorname{ord}_t y \geq n\}. \tag{1}$$

The same intersection results if one restricts to divisors with exactly k common centers with the zero reference chain. For every $j, q \geq 1$,

$$\overline{\operatorname{Cont}}^q(D_j) = \{\operatorname{ord}_t x \geq q,\ \operatorname{ord}_t y \geq jq\}. \tag{2}$$

This remains true for the closure of the at-least-q contact locus. Consequently, for $n = 5$, neither $I_{5,2}$ nor $I_{5,3}$ equals $\overline{\operatorname{Cont}}^q(D_j)$ for any positive integers q, j.

The decisive assertion is the last conclusion: the required reference divisor cannot be repaired by merely shifting its index. The general formula also supplies positive boundary checks. In particular $I_{n,k} = \overline{\operatorname{Cont}}^q(D_k)$ holds exactly when $k \mid n$, with $q = n/k$.

2 The entire free family

The source explicitly allows free as well as satellite prime divisors; the free/satellite sequence with proximity data encodes combinatorial type [1, pp. 5–7]. A point is free if it lies on the smooth locus of the exceptional divisor. No restriction to final divisors of minimal embedded resolutions of singular branches is imposed. Smooth curvettas and the extra free blowups after resolving them are included.

Lemma 2. *Every member of $\mathcal{F}_{n,k}$, and no additional divisor, is described by*

$$P(x) = a_k x^k + \cdots + a_{n-1} x^{n-1}, \qquad y = P(x) + x^n w, \qquad (a_k, \dots, a_{n-1}) \in \mathbb{C}^{n-k}. \tag{3}$$

Thus this is the full family, not a selected subfamily.

Proof. After j blowups along $y = 0$, the finite chart is $y = x^j u_j$. For $j \geq 2$, the newest exceptional divisor has equation $x = 0$; its sole intersection with older exceptional components is the point $u_j = \infty$. Blowing up a finite point $u_j = a$ gives the two charts

$$u_j = a + x u_{j+1}, \qquad x = vh, \quad u_j = a + h. \tag{4}$$

In the second chart the new component is $h = 0$, the old component is $v = 0$, and on the overlap $u_{j+1} = 1/v$. The one point omitted from the first chart is therefore precisely the satellite intersection, not an omitted free choice.

Starting with the fixed prefix, repeated substitution of the first formula in (4) gives exactly (3). Conversely every finite coefficient tuple chooses only free points. Different tuples first differing in one coefficient choose a different center before creation of the final divisor, and hence a different final divisor.

For completeness, the minimal sequence for one final prime divisor is its chain of predecessors. A center on an older component away from the newest component would make the intervening

blowup extraneous, so cannot occur in that minimal sequence. A satellite choice changes the proximity data, and therefore the specified all-free type. This excludes other apparent completions. Its weighted graph is the chain with $n-1$ weights -2 and final weight -1. The initial fixed tangent allows x to be the parameter of every smooth curvetta under discussion. □

For a nonzero prefix jet one first makes the polynomial coordinate change subtracting that jet from y; this gives the same statement in regular coordinates. The counterexample needs only the displayed zero reference. In the conditional exact-contact reading, exactly k common centers means $a_k \neq 0$ in (3).

3 Closures, including satellite limits

Lemma 3. *For the divisor E_P in (3), put $z = y - P(x)$. Then*

$$\overline{N}_{E_P} = S_P := \{\operatorname{ord}_t x \geq 1,\ \operatorname{ord}_t z \geq n\}. \tag{5}$$

This includes limiting arcs with $x = 0$, the constant arc, and lifts meeting a satellite point.

Proof. The coordinate change $(x, y) \mapsto (x, z)$ is polynomial and invertible. The equations of S_P are just the vanishing of the first $n-1$ coefficients of z. Its coefficient ring is a polynomial ring in the remaining variables, so S_P is a reduced irreducible cylinder. The nonempty open subset $\operatorname{ord}_t x = 1$ is dense. Every arc there has $w = z/x^n \in \mathbb{C}[[t]]$, so lifts in (3), meeting the newest component transversely away from its older intersection. Hence S_P is contained in the closure of the transverse locus.

For the reverse inclusion, the two charts covering the final component have maps

$$(x, w) \longmapsto (x, z = x^n w), \qquad (v, h) \longmapsto (x = vh,\ z = v^{n-1}h^n). \tag{6}$$

An arc centered on the new component has respectively $\operatorname{ord}_t x \geq 1$, or $\operatorname{ord}_t h \geq 1$; in either case $\operatorname{ord}_t x \geq 1, \operatorname{ord}_t z \geq n$. This proves containment even at its satellite intersection and hence (5).

One can check density directly without omitting degenerate arcs. For any $(x_0, y_0) \in S_P$, set

$$x_\epsilon = x_0 + \epsilon t, \qquad y_\epsilon = P(x_\epsilon) + y_0 - P(x_0).$$

The coefficients are polynomial in ϵ; the arc at zero is the given one, and away from at most one value of ϵ it lies in the transverse locus. This handles $x_0 = 0$ as well. A separate specialization for each P suffices for membership in each closure. □

Proof of (2). For D_j the charts are (6) with $P = 0$ and $n = j$. The contact order in the first chart is $\operatorname{ord}_t x$; in the second it is $\operatorname{ord}_t h$. Exact contact q, or contact at least q, therefore implies $\operatorname{ord}_t x \geq q, \operatorname{ord}_t y \geq jq$. Conversely the cylinder with those two inequalities is irreducible, and its nonempty dense open subset $\operatorname{ord}_t x = q$ lifts in the first chart with exact contact q. Thus the closures for both conventions are that cylinder. This argument works also for $j = 1$, whose two charts are the usual first-blowup charts. □

4 Intersections and uniform separation

Proof of (1) and the exact-contact variant. The members $P = 0$ and $P = x^k$ force $\operatorname{ord}_t y \geq n$ and $\operatorname{ord}_t x^k \geq n$. Since orders of nonzero formal series are integers, the latter is $\operatorname{ord}_t x \geq \lceil n/k \rceil$.

Conversely, these inequalities imply $\operatorname{ord}_t P(x) \geq n$ for every polynomial in (3), because all its exponents are at least k. This proves (1) for the entire family.

For the exact-contact family, $a_k \neq 0$. Its two members $P = x^k$ and $P = 2x^k$ force $\operatorname{ord}_t x^k \geq n$ by subtraction, and then $\operatorname{ord}_t y \geq n$. The converse is unchanged. This calculation treats a possible reading; it does not assert that the source defined its bare relation that way. □

Order cylinders with finite independent thresholds (a, b) and (a', b') are equal only if $a = a'$ and $b = b'$. For example, give one coordinate exactly its lower threshold and the other arbitrarily high order, and interchange their roles. Taking $n = 5$ gives

$$I_{5,2} = \{\operatorname{ord}_t x \geq 3, \operatorname{ord}_t y \geq 5\}, \qquad I_{5,3} = \{\operatorname{ord}_t x \geq 2, \operatorname{ord}_t y \geq 5\}. \tag{7}$$

Equality with (2) would require $(q, jq) = (3, 5)$ or $(2, 5)$. Neither $3j = 5$ nor $2j = 5$ has an integer solution. This proves the remaining conclusion of Theorem 1 for every q, j, without a bounded search.

Here are explicit convergent polynomial separating arcs, also fixing the direction of failure. Put $r = 3$ or 2. If $q < r$, the arc $(t^q, t^{\max(5,jq)})$ lies in (2) and outside (7). If $q > r$, (t^r, t^5) lies in (7) and outside (2). If $q = r$, use the latter arc when $jr > 5$, and use (t^r, t^{jr}) when $jr < 5$. The remaining case $jr = 5$ does not occur.

Explicit normalization	Intersection	Reference tested
Last-shared-divisor convention, prefix c_1, c_2	$I_{5,2}$	D_2
Zero-based $p_0 = c_1, p_2 = c_3$, contact at least 2	$I_{5,2}$	D_3
Fix the selected free c_3 as well	$I_{5,3}$	D_3
One-based $p_3 = c_3$, contact at least 3	$I_{5,3}$	D_3
Conditional exact contact 2 or 3	$I_{5,2}$ or $I_{5,3}$	Every D_j

In particular c_3 is genuinely free, and fixing it still leaves the two parameters for c_4, c_5. A three-blowup example can become a singleton after fixing its final center; the five-blowup example avoids that boundary issue. The raw transverse loci are not substituted for their closures.

5 Scope, prior framework and reproducibility

The result concerns the *specified reference multiple*. The cylinders in (7) are themselves maximal divisorial sets for the primitive toric valuations with weights $(3, 5)$ and $(2, 5)$. This follows from the established toric description [3, Example 2.10]; adjacent unimodular charts are respectively

$$(x, y) = (u^3 v, u^5 v^2), \qquad (x, y) = (u^2 v, u^5 v^3).$$

Their exponent matrices have determinant 1. The inverse monomials are

$$(u, v) = (x^2/y, y^3/x^5), \qquad (u, v) = (x^3/y, y^2/x^5).$$

The dense exact-order loci have $\operatorname{ord}_t u = 1$ and v a unit. Thus allowing a different arbitrary prime divisor removes this particular obstruction. No negative solution of the generalized Nash problem, no statement about Conjecture 2, and no new general theory of maximal divisorial sets is claimed.

We also distinguish closed sets from nonreduced intersections. For $n = 5, k = 2$, the two-member coefficient ideal after eliminating the first four coefficients of y is

$$(x_1^2,\ 2x_1x_2,\ x_2^2 + 2x_1x_3),$$

whose radical is (x_1, x_2), although x_1 is not in that ideal. For $k = 3$ the corresponding ideal is $(x_1^3, 3x_1^2x_2)$, with radical (x_1). Equation (7) states the reduced supports, not equality of these nonreduced schemes. Divisible cases $n = 6, k = 2, 3$ satisfy the reference equality, and fixing the final center gives the singleton D_n. Varying all tangents for an all-free length-n type gives $\{\mathrm{ord}_t\, x, \mathrm{ord}_t\, y \geq n\} = \overline{\mathrm{Cont}}^n(D_1)$, consistent with the first-exceptional specialization rather than refuting it.

The standard maximal-divisorial/contact framework is due to prior work, including [3]; combinatorial invariance and finite cylindrical intersections are proved in [1]. The contribution here is the explicit complete free-family calculation and its reference-ray obstruction. The source's authors still discuss their two conjectures as conjectures in [4, Section 8]. An extensive but bounded primary-source investigation is documented in the accompanying `SOURCE_AUDIT.md`. It located no authenticated earlier publication of this exact negative calculation. This is not a certificate of worldwide firstness, continuous openness, or completeness of the literature search. The exact claim is tied to the inspected arXiv1501.06898v3 and workshop2016 bodies. The publisher-typeset full text was not accessed; the original paper's DOI is a reference, not this counterexample's publication DOI. No author-approved erratum, intended new hypothesis, or unique resolution of its notation is asserted.

The source package includes an independently written standard-library checker, expected output, editable TeX, the reviewed PDF, source/provenance notes and reproduction instructions. Exact finite computations challenge coefficient substitutions, two-chart formulas, degeneracies, all listed finite witness cases, and boundary mutations. They supplement the universal proofs above; they do not certify arbitrary complex parameters, priority, or infinite-arc geometry. The package builder reproduces archives from the frozen PDF; it does not pretend to typeset that PDF.

AI and review disclosure. AI tools were extensively involved in mathematical research, proof development and reconstruction, symbolic and exact computation, literature/source investigation, manuscript preparation, editing and adversarial auditing. Alec Kriebel is the named human author. This manuscript is an unrefereed preprint and has not undergone formal independent human peer review or journal acceptance. Internal AI audits and exact supplementary computations do not provide those certifications or proof-assistant verification.

References

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