# Forward generation does not ensure GNS descent: an explicit continuous counterexample

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**Abstract**

For a general conditional-expectation dilation of a Markov semigroup, generation by the observed algebra and its forward translates does not ensure descent through the expectation's fixed GNS representation. We give a unital separable, injective, point-norm continuous example for the two-state exponential relaxation semigroup. Its GNS image is $M_2(\mathbb{C})$, and at every positive time an explicit nonzero positive element leaves the representation kernel. This refutes the concrete generation-only proposal in Skeide's 2009 module-approach problem. The clock/cocycle method is established: the negative phenomenon is also a short corollary of an earlier clock dilation of Burgarth, Facchi and Hillier. The note makes the connection explicit and supplies a small matrix realization with exact witnesses; it makes no claim of literature-first priority.

**Disclosure.** AI tools were used extensively in constructing, checking, auditing and writing this work. This is an unrefereed research note; its adversarial reviews were performed by AI tools and do not constitute human peer review.

## The question and the result

In the first module-approach problem of [1, pp. 542–543], Skeide proposes passing a general dilation $(A, \theta, i, p)$ through the GNS correspondence of $p$, and then extending its represented dynamics to all adjointable operators. Generation of $A$ by $i(B)$ and its time translates is offered as a sufficient "not too big" condition. We use the report's general definition [1, pp. 497–498]: $ip$ is a conditional expectation onto $i(B)$ and $p\theta_t i = T_t$. The printed direction of $p$ in the problem is reversed; the definition and diagrams specify $p : A \to B$. The stronger equation $p\theta_t = T_t p$ on all of $A$ is not assumed.

Let $\mathcal{D} \subset M_2(\mathbb{C})$ be the diagonal algebra, identified with $\mathbb{C}^2$, and let $\Delta : M_2(\mathbb{C}) \to \mathcal{D}$ extract the diagonal.

**Theorem 1.** *There exist a unital separable $C^*$-algebra $A$, a point-norm continuous semigroup $(\theta_t)_{t\geq 0}$ of injective unital *-endomorphisms, a unital embedding $i : \mathcal{D} \to A$, and a conditional expectation $ip : A \to i(\mathcal{D})$ such that*

$$A = C^*(\theta_r(i(\mathcal{D})) : r \geq 0), \qquad T_t = p\theta_t i, \qquad P_t = \frac{1}{2}\begin{pmatrix} 1+e^{-t} & 1-e^{-t} \\ 1-e^{-t} & 1+e^{-t} \end{pmatrix}$$

*is the transition matrix of $T_t$ on $\mathbb{C}^2$. For the left representation $\pi$ in the GNS correspondence of $p$,*

$$\pi(A) \cong M_2(\mathbb{C}), \qquad \theta_s(\ker \pi) \not\subseteq \ker \pi \quad \textit{for every } s > 0.$$

*Each failure has an explicit nonzero positive witness $a_s \in \ker \pi$. Consequently no semigroup on that represented algebra, or on all $\mathcal{B}^a(E)$ of the fixed GNS correspondence, can intertwine $\pi$ and $\theta$.*

## Construction and proof

Let $X = [-\infty, +\infty]$ be the two-endpoint compactification of $\mathbb{R}$ with its order topology. Define

$$g(x) = \begin{cases} 0, & x \le 0, \\ \frac{1}{2}\arccos(e^{-x}), & x > 0, \end{cases} \qquad g(-\infty) = 0, \quad g(+\infty) = \pi/4,$$

and put

$$R(z) = \begin{pmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{pmatrix}, \qquad V(x) = R(g(x)), \qquad u_t(x) = V(x)^* V(x+t).$$

Translation fixes the endpoints of $X$. It is jointly continuous in $(t, x)$: for time in a bounded neighborhood, sufficiently far tails remain in any prescribed endpoint neighborhood. Thus, on $C(X, M_2(\mathbb{C}))$,

$$\alpha_t(f)(x) = u_t(x) f(x+t) u_t(x)^*, \qquad t \in \mathbb{R},$$

defines a point-norm continuous automorphism group. Indeed $u_s(x)u_t(x+s) = u_{s+t}(x)$, and joint continuity on compact time intervals times $X$ gives uniform continuity for each fixed $f$. No differentiability of $g$ at zero is required.

View $\mathcal{D}$ as the constant diagonal functions and set

$$A = C^*(\alpha_r(\mathcal{D}) : r \ge 0), \quad \theta_t = \alpha_t|_A \ (t \ge 0), \quad i(b)(x) = b, \quad p(f) = \Delta(f(0)).$$

The shift of a generator at $r$ is the generator at $r + t$, so $A$ is forward invariant and the asserted generation equality is exact. Restrictions of the ambient automorphisms are injective unital endomorphisms. Norm continuity allows rational times to generate $A$, proving separability. Evaluation and $\Delta$ are unital completely positive; $pi = \mathrm{id}_{\mathcal{D}}$ and $p(i(b_1)fi(b_2)) = b_1 p(f) b_2$. Hence $ip$ is a conditional expectation.

At $x = 0$, $u_t(0) = R(g(t))$ for $t \ge 0$. Compressing a constant diagonal matrix gives weights $\cos^2 g(t) = (1 + e^{-t})/2$ and $\sin^2 g(t) = (1 - e^{-t})/2$. These are precisely $P_t$. Writing

$$J = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad K = \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \qquad P_t = J + e^{-t}K,$$

we have $J^2 = J$, $K^2 = K$, $JK = KJ = 0$ and $J + K = I$. This proves the Markov semigroup law for every real $t \ge 0$. The maps are UCP and uniformly continuous on $\mathbb{C}^2$.

**The actual GNS representation.** Let $q = \mathrm{diag}(1, 0)$ and $s_0 = \log 2$. Evaluation of $\theta_{s_0}(q)$ at zero is

$$Q = \begin{pmatrix} 3/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 1/4 \end{pmatrix}.$$

The evaluation image contains $q$, $1 - q$, $qQ(1-q) = (\sqrt{3}/4)e_{12}$ and its adjoint. Therefore $\mathrm{ev}_0(A) = M_2(\mathbb{C})$. The GNS correspondence is concretely

$$E = M_2(\mathbb{C}), \qquad z \cdot b = zb, \qquad \langle z, w \rangle = \Delta(z^* w), \qquad \xi = I_2, \qquad \pi(f)z = f(0)z.$$

To verify this, the map $a \otimes b \mapsto a(0)b$ on the algebraic GNS construction preserves its inner product. The concrete inner product is definite because its diagonal entries are column squared norms, and the evaluation image proves surjectivity and cyclicity. Finite dimension gives completeness. In particular,

$$\ker \pi = \{f \in A : f(0) = 0\}.$$

This is the full representation kernel, not an identification with a single vector-state null space in general. The two columns identify $\mathcal{B}^a(E) = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$; $\pi(A)$ is the diagonal copy. The module is full since $\langle \xi, \xi \rangle = 1_{\mathcal{D}}$.

**A positive witness at every positive time.** Fix $s > 0$, put $r = e^{-s} \in (0, 1)$ and $Q_s = \theta_s(q)(0)$, and define

$$a_s = q\theta_s(q)q - \frac{1+r}{2}q.$$

Then $a_s \in A$ and $a_s(0) = 0$. At $x = s$ its coefficient of $q$ is

$$\begin{aligned} d_s &= \cos^2(g(2s) - g(s)) - \frac{1+r}{2} \\ &= \frac{r^3 + (1-r^2)\sqrt{1+r^2} - r}{2} = \frac{(1-r^2)(\sqrt{1+r^2} - r)}{2} > 0. \end{aligned}$$

Here the angle-difference identity uses $\cos(2g(s)) = r$, $\cos(2g(2s)) = r^2$, and their positive sines. Consequently

$$\theta_s(a_s)(0) = d_s Q_s \neq 0.$$

Moreover $a_s$ is positive everywhere. For $x > 0$, $g'(x) = 1/(2\sqrt{e^{2x} - 1})$ is positive and decreasing. Comparing its integrals on $[x, x+s]$ and $[0, s]$ gives $0 \leq g(x+s) - g(x) \leq g(s)$. For $x \leq 0$ the same bound follows from monotonicity and $g(x) = 0$, and at the endpoints the increment is zero. Since $\cos^2$ decreases on $[0, \pi/4]$,

$$a_s(x) = \left(\cos^2(g(x+s) - g(x)) - \cos^2 g(s)\right) q \geq 0.$$

Strict positivity of $d_s$ proves $a_s \neq 0$. At $s = \log 2$, $d_s = 3(\sqrt{5} - 1)/16$, giving an exact radical witness. An intertwining map would send both $\pi(0)$ and $\pi(a_s)$ to the same value while their required images are 0 and $d_s Q_s$. This proves the theorem.

## Antecedents and the finite-image obstruction

The point-localized clock and translated-unitary construction are established in the cited arXiv v2 of Burgarth–Facchi–Hillier (BFH) [2, Definition 4.1, Theorem 4.4, Step 2, equation (11)]. Finite time-dependent unitary dilations and their initial square-root singularities are treated by Dive–Mintert–Burgarth [3, Method, Appendices A–B, arXiv v3]. Their first-derivative obstruction has a bounded-total-Hamiltonian formulation in the cited arXiv v3 of [4, Theorem 1]. The following elementary conditional-expectation version explains the connection; it is proved here rather than attributed to those papers as a fixed-GNS theorem.

**Proposition 2.** *For a point-norm continuous general unital dilation of a norm-continuous Markov semigroup on a finite-dimensional algebra $B$, if $\pi(A)$ is finite dimensional and $\ker \pi$ is invariant at every time, then the generator of the compressed semigroup is a *-derivation on $B$.*

*Proof.* Kernel invariance gives a continuous endomorphism semigroup $\beta_t$ on $C = \pi(A)$. Finite dimension makes continuity operator-norm continuity. For small $h > 0$, $S = \int_0^h \beta_u \, du$ is invertible because $S/h \to I$. The identity $\beta_t S = \int_t^{t+h} \beta_u \, du$ proves right differentiability at zero. Its generator $\delta$ is a *-derivation by differentiation of multiplicativity. The embedding $j = \pi i$ is faithful, and $\bar{p}(c) = \langle \xi, c\xi \rangle$ is a $B$-bimodule UCP map with $\bar{p}j = \mathrm{id}_B$. Thus, identifying $B$ with $j(B)$,

$$L(bc) = \bar{p}(\delta(bc)) = \bar{p}(\delta(b))c + b\bar{p}(\delta(c)) = L(b)c + bL(c).$$

Adjoints are preserved as well. □

In fact BFH's earlier explicit amplitude-damping clock already yields the negative generation-only conclusion as a short corollary. The model is in Section 2, equations (1)–(3), of the cited arXiv v2 of [2]. Its finite unitary path has angle

$$G(x) = \mathrm{sgn}(x) \arccos(e^{-|x|/2}), \qquad G(0) = 0,$$

with finite endpoint limits. It therefore extends to $[-\infty, +\infty]$, where its translated cocycle is point-norm continuous. Restriction to the forward-generated observable algebra preserves the conditional expectation and the compression. Evaluation followed by finite bath compression gives a finite-dimensional GNS image. The decaying-level projection satisfies $T_t(q) = e^{-t}q$, whereas a derivation satisfies $qL(q)q = 0$. Proposition 2 rules out invariance at all times. The generic strong continuity of BFH's Hilbert-space unitaries alone would not justify point-norm continuity on all $C_b(\mathbb{R})$; the endpoint check for this explicit path is essential.

Thus the general negative phenomenon is not a new clock-dilation mechanism. Theorem 1 gives a small $M_2/\mathbb{C}^2$ realization and positive witnesses at every positive time. Bounded primary-source searches through 13 October 2026 found no publication explicitly stating the exact OWR fixed-GNS counterexample, but neither that search nor the explicit refinement certifies first priority. Full source locations, hashes and the prior-result deduction are in the verification package.

## Scope and reproducibility

This answers negatively the concrete forward-generation-only proposal in [1]. Failure of its first step refutes the proposed two-step conjunction. It also defeats any sufficient condition logically implied by forward generation alone. Useful stronger or incomparable minimality conditions, and extension after kernel invariance is separately imposed, remain outside the result. The related fixed-action question in [5, Section 5] should be read with these same limits.

The example is a general C*-dilation with a nonfaithful expectation. It is not strong: $p(a_s) = 0$ but $p\theta_s(a_s) \neq 0$. It is not a weak/corner dilation, and no normal von Neumann version or nonexistence of a different dilation of the same Markov semigroup is asserted. These categories are distinguished in the cited arXiv v3 of [6]: Definition 2.3, diagram (2.3), and Appendix R(i), pp. 211–212. The locator refers specifically to that preprint version.

The accompanying package contains this source and PDF, the frozen candidate, independent exact arithmetic checks, analytic audits, priority evidence and a hash-checked replay program. Finite computations supplement the all-real-time proof; they do not establish its continuous quantifiers. Hashes certify byte integrity, not mathematical truth. No third-party full texts are redistributed.

## References


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