An exact period criterion for focal-distance products in elliptic billiards

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Abstract

For a noncircular elliptic billiard with a strictly nested nondegenerate confocal elliptical caustic, we prove that the product of ordinary distances from all vertices to either fixed focus is independent of phase if and only if the least period is 2 modulo 4. This proves the least-period interpretation of entry k_{114} in Reznik–Garcia–Koiller's list of unproved invariants and classifies every other primitive period and every repeated traversal. The proof identifies the complete divisor of the focal-distance factor in the classical Jacobi parameter. An exact doubled-triangle example shows why a condition on return length alone is insufficient. The parametrization, identities and divisor method are established tools. This AI-assisted research note is unrefereed.

1 The invariant and its exact scope

Let the boundary and caustic be

$$\mathcal{E}: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \mathcal{C}: \frac{x^2}{A^2}+\frac{y^2}{B^2}=1, \quad a>b>0, \quad A>B>0,$$

with $a>A$, $b>B$ and $a^2-b^2=A^2-B^2=c^2>0$. Fix a periodic billiard family tangent to $\mathcal{C}$, with *least period* n, vertices P_j, and fixed foci $F_\pm=(\pm c,0)$. Write

$$D_n^{\pm}=\prod_{j=0}^{n-1}|P_j-F_\pm|.$$

These are ordinary positive Euclidean distances.

Reznik, Garcia and Koiller recorded D_N^+ as invariant k_{114} for $N\equiv 2 \pmod 4$, with proof column *?*, in their 2020 preprint and 2021 article [1, Table 2, p. 345]. Their introductory setting is a confocal ellipse pair. The entry does not explicitly settle whether N is least period or a return length permitting repetitions. The following theorem decides both interpretations within that setting.

Theorem 1. *For the fixed noncircular confocal pair above, each $D_n^{\pm}$ is independent of phase if and only if $n\equiv 2 \pmod 4$. Every admissible primitive star winding and both orientations are included. For every even n, $D_n^+=D_n^-$ at each phase. For an r-fold traversal, $r\geq 1$, the product is $(D_n^{\pm})^r$, and is constant exactly in the same least-period class. For a circular boundary the two foci coincide with the center, and every listed product is a^N.*

For return length $N\equiv 2 \pmod 4$, its least period divides N and is either odd or 2 modulo 4. The product is therefore constant exactly when its least period is even. Hyperbolic and degenerate caustics are outside the source setting and this theorem.

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2 The classical real parameter

Put $k = c/A \in (0,1)$, $k' = \sqrt{1-k^2}$, $K = K(k)$ and $K' = K(k')$. All Jacobi functions have *modulus* k. Stachel's canonical parametrization [2, Theorem 4.3 and (4.9), p. 1614] is

$$P(u) = (-a\,\mathrm{sn}\,u, b\,\mathrm{cn}\,u), \quad a = A\frac{\mathrm{dn}\,v}{\mathrm{cn}\,v}, \quad b = \frac{B}{\mathrm{cn}\,v}, \quad 0 < v < K, \quad P_j(u) = P(u+2jv). \tag{1}$$

Here v is uniquely determined by $\mathrm{cn}\,v = B/b$; confocality gives the other axis relation.

For completeness, the geometric bridge can be checked directly. The tangent at $Q(U) = (-A\,\mathrm{sn}\,U, B\,\mathrm{cn}\,U)$ on $\mathcal{C}$ is

$$-x\,\mathrm{sn}\,U/A + y\,\mathrm{cn}\,U/B = 1.$$

Jacobi addition [3, Section 22.8] places both $P(U-v)$ and $P(U+v)$ on it: substitution gives

$$\frac{\mathrm{dn}^2\,v\ \mathrm{sn}^2\,U + \mathrm{cn}^2\,U}{1 - k^2\,\mathrm{sn}^2\,U\ \mathrm{sn}^2\,v} = 1.$$

The contact lies strictly inside $\mathcal{E}$, so this is an actual tangent chord. The two tangent directions at $P = (x,y)$ satisfy $d^{\mathsf{T}} M d = 0$, with

$$M = \begin{pmatrix} B^2 - y^2 & xy \\ xy & A^2 - x^2 \end{pmatrix}.$$

The outer normal $\nu = (x/a^2, y/b^2)$ and its perpendicular $t = (-y/b^2, x/a^2)$ obey

$$\nu^{\mathsf{T}} M t = \frac{xyc^2}{a^2 b^2}\left(1 - x^2/a^2 - y^2/b^2\right) = 0.$$

Also $\det M = A^2B^2(1 - x^2/A^2 - y^2/B^2) < 0$. Thus the distinct inward tangent rays are interchanged by reflection across the normal, proving the elastic billiard law. The real Jacobi amplitude increases strictly through 2π over $4K$; hence this parameter covers every boundary point and both oriented caustic-tangent families.

Real injectivity modulo $4K$ gives the least-period condition

$$\delta = 2v = \frac{4K\tau}{n}, \qquad \gcd(\tau, n) = 1, \qquad 0 < \tau < n/2. \tag{2}$$

The orientation is positive and τ is its turning number. Neither $n = 1$ nor $n = 2$ can occur with $0 < v < K$. Elementary ellipse algebra gives

$$|P(u) - F_{\pm}|^2 = (a \pm c\,\mathrm{sn}\,u)^2.$$

Since $a > c$, the linear factors are strictly positive on the real axis. Therefore the actual distance functions are

$$g_{\pm}(u) = a \pm c\,\mathrm{sn}\,u, \qquad D_n^{\pm}(u) = \prod_{j=0}^{n-1} g_{\pm}(u + j\delta). \tag{3}$$

3 The complete divisor and the period criterion

Regard the focal factor on

$$T = \mathbb{C}/(4K\mathbb{Z} + 2iK'\mathbb{Z}), \qquad p = iK'.$$

This torus is used for sn and $g_\pm$; only the real vertex period $4K$ was used in (2). The standard Jacobi periods, poles and shifts [3, Tables 22.4.1 and 22.4.3] give two simple poles of g_+, at $p, p+2K$, and

$$\operatorname{sn}(p + K \pm v) = \frac{\operatorname{dn} v}{k \operatorname{cn} v} = \frac{a}{c}, \qquad \operatorname{sn}(u + 2K) = -\operatorname{sn} u.$$

Thus its zeros are $q_\pm = p - K \pm v$. They are distinct and nonpolar because $0 < v < K$. The coefficient $c \neq 0$ prevents removal of either pole. An elliptic function has equal total zero and pole degree, so these two zeros are simple and exhaustive.

Let $H = \langle \delta \rangle$ on T. It has exact order n. Subtracting p and dividing real offsets by $4K$ identifies it with $\{j/n \bmod 1\}$. Orders in a product add, so the complete product divisor is the sum of translates of the divisor of g_+. The two zeros differ by one step δ; their translates make a single zero coset of multiplicity two, represented by

$$z = -\frac{1}{4} + \frac{\tau}{2n}.$$

The pole cosets are H and $1/2 + H$.

If n is odd, these pole cosets are disjoint. Both

$$nz = \frac{-n + 2\tau}{4}, \qquad n(z - 1/2) = \frac{-3n + 2\tau}{4}$$

have odd numerators and are not integers. The zero coset misses both pole cosets, leaving $2n$ simple poles.

If n is even, τ is odd and $1/2 \in H$. The pole cosets coincide, giving n double poles. The double-zero coset cancels them precisely when

$$nz = \frac{-n + 2\tau}{4} \in \mathbb{Z},$$

equivalently $n \equiv 2 \pmod 4$. Then the full divisor vanishes, the product extends holomorphically to the compact connected torus, and it is constant. For $n \equiv 0 \pmod 4$, it retains n double poles.

The products are holomorphic near the real axis. If a product with uncancelled poles were constant on the real family, the identity theorem would force its meromorphic extension to be constant, a contradiction. This proves physical real-family nonconstancy: (3) equals the ordinary Euclidean product itself on that line.

Finally $g_-(u) = g_+(u + 2K)$. The cyclic product is invariant under H, proving the same classification for both foci; when n is even, $2K \in H$ also proves equality at each phase. For $n = 2m \equiv 2 \pmod 4$, antipodal pairing and phase-zero evaluation give the positive value

$$D_n^\pm = \prod_{j=0}^{m-1} \left(a^2 - c^2 \operatorname{sn}^2(2jv)\right) > 0.$$

An r-fold traversal raises the positive primitive product to the power r. This operation is injective on positive reals, so it preserves constancy in both directions. For a circle every focal distance is a. This proves Theorem 1.

4 An exact return-length counterexample

Take $a = \sqrt{15}$, $b = \sqrt{7}$, $c = \sqrt{8}$, $A^2 = 135/16$, $B^2 = 7/16$. The triangle

$$T = \big((a, 0), (-3a/4, 7/4), (-3a/4, -7/4)\big)$$

and its central reflection are primitive three-bounce members of one confocal family. Their vertices lie on $\mathcal{E}$. The side lines are $x = -3a/4 = -A$ and $x \pm ay = a$; tangency follows from $A^2 + a^2B^2 = a^2$. At the upper off-axis vertex the inward unit rays $(a/4, -1/4)$, $(0, -1)$ sum to five times the inward normal $(a/20, -1/4)$. Symmetry handles the other vertices, independently checking physical reflection.

At $F_+ = (c, 0)$ the primitive products are

$$(a - c)(a + 3c/4)^2 = \frac{15a}{2} + 3c, \qquad (a + c)(a - 3c/4)^2 = \frac{15a}{2} - 3c.$$

Twice traversing each triangle gives return length six, but the six-event products differ by

$$\left(\frac{15a}{2} + 3c\right)^2 - \left(\frac{15a}{2} - 3c\right)^2 = 90ac = 180\sqrt{30} > 0.$$

Thus listed return length cannot replace least period.

5 Attribution, verification and status

The theorem proves the least-period k_{114} observation and classifies its broader return-period interpretation, which is partly false. Stachel's parametrization, classical Jacobi identities and pole-removal method are prior machinery. Meromorphic Poncelet functions and pole-removal arguments were used by Schwartz [6] and, for example, Akopyan–Schwartz–Tabachnikov [4] and Roitman–Garcia–Reznik [5]. The target-specific step is the focal divisor and its coset arithmetic; no new elliptic-function method is claimed. Ferudun's related unrefereed note [7] treats the different outer-tangent-polygon product k_{115}.

The accompanying priority note records the primary sources checked and search limits. No explicit earlier proof of this exact target was located in the checked sources; the novelty of this specific formulation and exclusive historical priority are not certified. The full text of a related inertia-and-billiards paper by Dragović and Gajić [8] was unavailable; its uninspected billiard sections remain a disclosed bibliography coverage gap. Portable exact checks reproduce finite divisor inventories and physical counterexample arithmetic. They support the written proof, and do not replace its all-period analytic argument. No validated numerical certification or proof-assistant formalization is claimed.

AI tools were used extensively in solving, drafting, comparing sources, reproducing computations and independent adversarial reviews. This preprint has not undergone conventional human peer review or refereeing. The author is responsible for its claims.

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