# A fixed moment separator without a rational normalized sum-of-squares certificate

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**Abstract**

We apply Scheiderer's classical rational quartic, which is a sum of real squares but not rational squares, to the fixed-polynomial certificate question in Oberwolfach Report 14/2023. On the rational unit ball in three variables, an explicit rational degree-four moment vector has mass one and annihilates a rational polynomial $p$ with minimum one. The identity $p \in 1+Q_{\mathbb{R}}(g)$ has a real certificate, whereas $p \notin 1 + Q_{\mathbb{Q}}(g)$, even allowing arbitrary finite multiplier degrees. Here rational certificates mean rational polynomial square factors. The all-degree obstruction is a classical local leading-form argument. The moment matrix is not positive semidefinite, another polynomial gives a rational certificate for the same vector, and every rational rescaling $cp$, $c > 1$, has a rational normalized certificate. Thus the example exhibits the phenomenon for one fixed boundary separator; it does not address the companion paper's strict-margin regime.

## 1 The question and certificate convention

The end of Naldi's contribution, joint work with Henrion and Safey El Din, in Oberwolfach Report 14/2023 [1, pp. 802–804] asks whether a rational polynomial certifying truncated-moment nonrepresentability can have no rational certificate for its fixed membership in $1 + Q(g)$. We exhibit this phenomenon under the rational-square convention below. The quartic, its irrationality mechanism, and the local obstruction used here are established results. The purpose of this note is their explicit application to the printed moment question and clarification of its normalization boundary.

For $F = \mathbb{Q}$ or $\mathbb{R}$, $X = (x, y, z)$, and $g \in F[X]$, define

$$Q_F(g) = \left\{ \sum_{i=1}^{r} a_i^2 + g \sum_{j=1}^{s} b_j^2 : r, s \geq 0, \quad a_i, b_j \in F[X] \right\}.$$

All sums are finite; there is no degree bound on the factors. A *rational certificate* means that these factors lie in $\mathbb{Q}[X]$. This is equivalent to rational positive-semidefinite Gram data [10, Lemma 1.6]. It is stronger than requiring rational coefficients only in the real-SOS multiplier polynomials. The printed OWR question does not formally specify this distinction; its Scheiderer reference provides the rational-SOS context. We state our convention explicitly rather than attribute an unstated formal definition to that source.

For $m = (m_\alpha)_{|\alpha| \leq 4}$, let

$$L_m \left( \sum_{|\alpha| \leq 4} p_\alpha X^\alpha \right) = \sum_{|\alpha| \leq 4} p_\alpha m_\alpha.$$

A representing measure is a nonnegative Borel measure supported on $K = \{X \in \mathbb{R}^3 : g(X) \geq 0\}$, with the specified moments.

## 2 Explicit example and proof

Use the classical Scheiderer quartic [3, Theorem 2.1 and Example 2.8]:

$$\begin{aligned} f = x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z \\ + 2x^2z^2 + xz^3 + yz^3 + z^4. \end{aligned} \tag{1}$$

Set

$$g = 1 - x^2 - y^2 - z^2, \qquad p = 1 + f, \qquad m_{000} = 1, \quad m_{400} = -1, \quad m_\alpha = 0 \text{ otherwise.} \tag{2}$$

There are $\binom{7}{3} = 35$ moment entries.

**Theorem 1.** *For the data (2), $K$ is the nonempty compact unit ball, and $Q_{\mathbb{Q}}(g)$ is Archimedean. The vector $m$ has no representing measure, even on $\mathbb{R}^3$, and $L_m(p) = 0$. Moreover,*

$$\min_K p = 1, \qquad p \in 1 + Q_{\mathbb{R}}(g), \qquad p \notin 1 + Q_{\mathbb{Q}}(g).$$

*The last nonmembership holds for every finite choice of polynomial square factors, of arbitrary degrees.*

We recall the credited quartic argument to make the obstruction checkable. No new quartic or Galois construction is claimed.

**Lemma 1** (Scheiderer's quartic)**.** *The polynomial $f$ in (1) is a sum of real polynomial squares and is not a sum of rational polynomial squares.*

*Proof.* Let $h(t) = t^4 - t + 1$. It has no real roots: for $t \leq 0$ or $t \geq 1$ its value is at least one, and for $0 < t < 1$ both $t^4$ and $1 - t$ are positive. Modulo 2, $h$ is irreducible: it has no linear factor and is not divisible by the sole irreducible monic quadratic $t^2 + t + 1$. Modulo 3,

$$h(t) = (t + 1)(t^3 - t^2 + t + 1),$$

where the cubic is irreducible and the factors are distinct. The usual factorization-cycle theorem gives a 4-cycle and a 3-cycle in the Galois group. Its order therefore is 12 or 24; the index-two subgroup $A_4$ contains no 4-cycle. Thus the group is $S_4$.

For the four roots $\alpha_i$, put $\ell_i = x + \alpha_i y + \alpha_i^2 z$. The norm identity $f = \prod_i \ell_i$ follows by expansion, or by the determinant of multiplication by $x + ty + t^2 z$ in $\mathbb{Q}[t]/(h)$. Every three $\ell_i$ are independent by the Vandermonde determinant. Complex conjugation pairs the roots. Choosing one root from each pair, $f = |\ell_1 \ell_2|^2$, after relabeling, is a sum of two real quadratic squares.

Suppose $f = \sum_j q_j^2$ with rational homogeneous quadratics $q_j$. The intersection of each conjugate pair of projective lines $\{\ell_i = 0\}$ is a real projective point, since its one-dimensional complex vector space is conjugation-stable. At those points $f = 0$, so every $q_j$ vanishes. Rational coefficients and the $S_4$ action then force vanishing at all six pairwise line intersections. On each of the four lines there are three distinct such points. A binary quadratic vanishing at three distinct projective points vanishes identically on that line. Hence all four distinct $\ell_i$ divide each $q_j$, impossible for a nonzero quadratic.

An inhomogeneous rational-square representation gives no escape. The highest homogeneous square terms cannot cancel over $\mathbb{R}$, so all factors have degree at most two. The constant part of $f$ forces all constant factors to vanish; its degree-two part then forces their linear parts to vanish. The factors would be precisely the homogeneous quadratics just excluded. □

For an explicit real certificate, let $\beta \in (-2,-1)$ satisfy $\beta^3 - 4\beta - 1 = 0$. Scheiderer's displayed identity is

$$\begin{aligned} U &= 2x^2 + \beta y^2 - yz + (2+\beta^{-1})z^2, \\ V &= 2xy - \beta^{-1}y^2 + 2\beta^{-1}xz + \beta yz - z^2, \\ 4f &= U^2 - \beta V^2. \end{aligned} \tag{3}$$

Existence of this negative root follows from opposite signs at $-2, -1$. Thus $f = (U/2)^2 + (\sqrt{-\beta}\, V/2)^2$. The supplement checks (3) exactly after clearing denominators, modulo the cubic.

*Proof of Theorem 1.* The geometry of $K$ is immediate, and its displayed rational ball generator makes the module Archimedean. The negative fourth moment $m_{400}$ excludes every nonnegative representing measure. The coefficient of $x^4$ in $f$ is one, so $L_m(p) = 1 - 1 = 0$. Lemma 1 gives $f \geq 0$ and $p \in 1 + Q_{\mathbb{R}}(g)$. Since $f(0) = 0$, the minimum is one.

For the remaining assertion, suppose a finite rational identity

$$f = \sum_i a_i^2 + (1 - x^2 - y^2 - z^2) \sum_j b_j^2$$

holds, with arbitrary factor degrees. At the origin, both weights are one, so every $a_i(0), b_j(0)$ vanishes. The homogeneous degree-two part is then the sum of the squares of all linear parts. It is zero, so these linear parts also vanish. Writing $a_i^{[2]}, b_j^{[2]}$ for the homogeneous quadratic parts, the degree-four part of the identity is exactly

$$f = \sum_i (a_i^{[2]})^2 + \sum_j (b_j^{[2]})^2.$$

The product of $g - 1$ with any $b_j^2$ starts at degree six; higher factor terms do not affect degree four. This contradicts Lemma 1. No global degree bound or assertion that higher weighted terms cannot cancel was used. □

This local argument is classical. Scheiderer's leading-form lemma and positive-generator completion argument already give its mechanism [4, Lemma 1.1 and proof of Proposition 6.1]. Benoist explicitly observes that this same quartic is not a sum of squares in $\mathbb{Q}[[x, y, z]]$ [5, Remark 2.7]. Alternatively, a rational ball-module identity would become a rational formal-square identity after multiplying the $b_j$ by the binomial series $\sqrt{1 - x^2 - y^2 - z^2}$. Nie previously applied the corresponding leading-form obstruction over the reals to the homogeneous Motzkin form on a ball [6, Example 5.3].

## 3 Normalization, scope, and related work

The following controls are part of the result, not optional qualifications. First, the same $m$ has the rationally certifiable separator

$$q = 1 + x^4, \qquad L_m(q) = 0, \qquad q - 1 = (x^2)^2.$$

Second, Powers' rational Putinar theorem for a rational ball generator [9, Theorem 7] gives $p \in Q_{\mathbb{Q}}(g)$: $p$ itself is strictly positive on $K$. For every rational $c > 1$,

$$cp - 1 = (c - 1) + cf \geq c - 1 > 0 \quad \text{on } K,$$

so the same theorem gives $cp \in 1 + Q_{\mathbb{Q}}(g)$, with $L_m(cp) = 0$. The obstruction concerns $c = 1$. For $c < 1$, $cp(0) - 1 < 0$, so even real normalized membership fails. No degree bound for Powers' certificates is asserted.

Third, $M_2(m)$, indexed by monomials of degree at most two, is not positive semidefinite: its diagonal entry indexed by $x^2$ is $m_{400} = -1$. The construction does not address a PSD-input restriction. Finally, a merely rational-coefficient list of real-SOS multipliers already exists, namely $\sigma_0 = f, \sigma_1 = 0$. It is the rational square factors that are impossible.

The companion preprint of Henrion, Naldi, and Safey El Din [2, Definition 3, Corollary 2, and Remark 2] distinguishes a strict-margin sufficient regime, $\min_K p > 1$. Our minimum-one example addresses the literal fixed-polynomial OWR question and is outside that corollary's regime. It supplies neither an algorithmic claim nor a minimal-relaxation-degree claim, and does not settle general Archimedean descent variants.

The same quartic's irrational Gram points were studied by Henrion, Naldi, and Safey El Din [7, §6.2]. Naldi and Sinn already used it to construct a strongly infeasible rational semidefinite program without a rational PSD separating certificate [8, Example 3.8]. That matrix-certificate application is substantial prior work; the present note uses the fixed normalized moment identity and arbitrary polynomial multiplier degrees instead. Our contribution is this explicit application and its scope clarification, rather than a new irrationality mechanism.

A dated bounded priority note is included in the supplement. It records the primary passages inspected and exact access limits, including the unexamined remainder of the 2026 corrected journal proof on descent [11], whose full body was unavailable. Its inspected univariate, strict-positive, and zero-dimensional-support scopes do not conflict with this example. The module here is already rationally Archimedean, and its support ideal is zero because $K$ has interior. The quotient $\mathbb{Q}[X]/(Q_{\mathbb{Q}}(g) \cap -Q_{\mathbb{Q}}(g))$ therefore has dimension three. No statement about the unavailable Appendix B theorem is inferred. We make no worldwide-firstness or current-openness assertion.

**Verification and disclosure.** The portable supplement contains this manuscript source, exact rational data, a standard-library checker and its full expected output, scope and priority notes, and a hash manifest. Its 2,059 finite checks cover the norm identity, finite-field factorizations, real-SOS identity, moment data, and representative leading-form/incidence controls. The proofs above, not finite searches, establish the Galois and all-degree nonmembership assertions. AI tools were used extensively in developing the application, literature comparison, exact computations, writing, and adversarial checking. This is an unrefereed preprint; those checks do not constitute independent human peer review or formal proof-assistant certification.

## References


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