A fixed birational map without a prime-field cycle-distribution limit

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Abstract

For the fixed birational map $L(u,v) = (u, (u^2+1)v)$, reduction at every prime $p \equiv 3 \pmod 4$ permutes all of $\mathbf{F}_p^2$. Its point-weighted distribution of periods divided by p has no limit. At every fixed $1/2 \le x < 1$, the distribution function is $1-\varphi(p-1)/(p-1)+o(1)$, with limsup 1 and liminf at most 23/24. Moreover, neither these distribution-function values nor the period measures converge on any further prime subset of positive upper relative density. The latter statement uses the classical continuous empirical law for $\varphi(p-1)/(p-1)$. A supplementary proposition treats all odd primes on the invariant regular domain, and under uniform-point deletion or arbitrary exceptional-period completion. These conventions cannot restore convergence on any positive-upper-density prime selection. The contribution is a precisely delimited application to the broad existence assertion in the 2004 Oberwolfach problem on finite-field maps. The triangular family, primitive-value estimates, totient mean and scalar distribution law are established ingredients; no new arithmetic theorem or historical firstness is claimed.

1 Statement and scope

Let

$$L(u,v) = (u, (u^2+1)v), \qquad L^{-1}(u,v) = \left(u, \frac{v}{u^2+1}\right). \tag{1}$$

These formulas define inverse rational maps over $\mathbf{Q}$ on a dense open set. The forward map is polynomial, but its inverse is not polynomial over $\mathbf{C}$. The rational invariant $I(u,v) = u$ has genus-zero generic fibers, and $L^m(u,v) = (u, (u^2+1)^m v)$ shows that L has infinite order. The map belongs, after exchanging the coordinates, to the older triangular polynomial family in Ostafe–Shparlinski [2, §2.1, (1)–(4)]: take $m = 1$, $a = 1$, $b = 0$, $g_0(X_1) = X_1^2 + 1$, and $h_0 = 0$.

Set $\mathcal{S} = \{p \text{ prime} : p \equiv 3 \pmod 4\}$. For $p \in \mathcal{S}$, write $T_p(z)$ for the least positive period of an affine point and define

$$\mu_p = \frac{1}{p^2} \sum_{z \in \mathbf{F}_p^2} \delta_{T_p(z)/p}, \qquad D_p(x) = \mu_p([0,x]) = \frac{1}{p^2} \#\{z : T_p(z) \le px\} \quad (x \ge 0). \tag{2}$$

For a prime set $\mathcal{A} \subseteq \mathcal{S}$, positive upper relative density in $\mathcal{S}$ means

$$\limsup_{X \to \infty} \frac{\#\{p \le X : p \in \mathcal{A}\}}{\#\{p \le X : p \in \mathcal{S}\}} > 0. \tag{3}$$

Since $\mathcal{S}$ has density $1/2$ among primes, this is equivalent to positive upper density relative to all primes.

Theorem 1. *The map (1) permutes all of $\mathbf{F}_p^2$ for every $p \in \mathcal{S}$. Put $\theta_p = \varphi(p-1)/(p-1)$.*

1. *For each fixed $x \in [1/2, 1)$, eventually along $\mathcal{S}$,*

$$|D_p(x) - (1 - \theta_p)| \le \frac{1}{p} + 64p^{-1/4}. \tag{4}$$

2. *For every such fixed x,*

$$\limsup_{\substack{p\to\infty \\ p\in\mathcal{S}}} D_p(x) = 1, \qquad \liminf_{\substack{p\to\infty \\ p\in\mathcal{S}}} D_p(x) \le \frac{23}{24}. \tag{5}$$

3. *If $\mathcal{A} \subseteq \mathcal{S}$ satisfies (3), then $D_p(x)$ does not converge as $p \to \infty$ in $\mathcal{A}$, for any fixed $x \in [1/2, 1)$. The probability measures μ_p do not converge weakly along $\mathcal{A}$ either.*

Vivaldi's 2004 contribution, joint work with Roberts [1, pp.2944–2945, Conjecture 1], uses the point weighting and scale in (2). It fixes the field-extension degree, permits suitable positive-density reduction primes, and then takes $q = p$. The stated existence assertion covers arbitrary birational maps; genus one enters a subsequent heuristic. Here the fixed degree is one, the forward polynomial acts on the full affine plane, and both directions are defined at every point for $p \in \mathcal{S}$. Part (iii) also rules out restoring the limit by a further positive-upper-density selection inside this admissible class. Proposition 4 below separately treats positive-density selections outside this class under explicitly specified deletion/completion conventions; it does not relabel the full-affine-permutation assertion as an all-prime theorem. Failure at $x = 3/4$ avoids the reciprocal integer thresholds in the proposed integrable step law. We draw no conclusion about narrower elliptic-fiber or polynomial-automorphism questions, or about the different prime-averaged statistic at scale p^2 in the source's later Conjecture 2.

2 Exact fiber dynamics and primitive multipliers

For $p \in \mathcal{S}$, -1 is a nonsquare, so $a_u = u^2 + 1 \neq 0$ for every $u \in \mathbf{F}_p$. On each vertical fiber, L multiplies the second coordinate by a_u. This proves the permutation assertion. Every $(u, 0)$ has period one; every (u, v) with $v \neq 0$ has period $\operatorname{ord}_{\mathbf{F}_p^\times}(a_u)$. In particular all points lie on cycles and all periods divide $N = p - 1$.

Define the number of primitive multipliers by

$$P_p = \#\{u \in \mathbf{F}_p : \operatorname{ord}_{\mathbf{F}_p^\times}(u^2 + 1) = N\}.$$

Every proper divisor of N is at most $N/2 < p/2$. For a fixed $x \in [1/2, 1)$ and $p > 1/(1-x)$, $N > px$. Thus exactly the primitive fibers contribute excluded points, $p - 1$ points per fiber:

$$D_p(x) = 1 - \frac{(p-1)P_p}{p^2}. \tag{6}$$

This counts points, rather than assigning equal weight to cycles. The threshold condition is essential: at $x = (p-1)/p$ all points are included. Also $D_p(0) = 0$ and $D_p(1) = 1$. No estimate uniform for moving $x \to 1$ is asserted.

Lemma 2. *For every $p \in \mathcal{S}$, with $\omega(N)$ the number of distinct prime divisors of N,*

$$\left| \frac{P_p}{p} - \theta_p \right| \le \frac{\theta_p(2^{\omega(N)} - 1)}{\sqrt{p}}. \tag{7}$$

Proof. We give the familiar character argument to specify the input completely. This mechanism is classical; a general quadratic primitive-value formulation appears in Booker–Cohen–Sutherland–Trudgian [3, §2, (5)–(6), Lemma 1]. Extend all multiplicative characters by zero at zero, including the principal character. For $a \in \mathbf{F}_p^\times$, the primitive-element indicator is

$$\mathbf{1}_{\mathrm{ord}(a)=N} = \theta_p \sum_{d \mid N} \frac{\mu(d)}{\varphi(d)} \sum_{\mathrm{ord}(\chi)=d} \chi(a). \tag{8}$$

Indeed, write $a = g^k$ for a generator g. For a prime $\ell \mid N$, the sum over characters of order ℓ is $\ell - 1$ if $\ell \mid k$ and -1 otherwise. The squarefree-divisor sum factors into $\prod_{\ell \mid N}(1 - c_\ell(k)/(\ell-1))$. Multiplication by θ_p gives zero if some $\ell \mid k$ and one otherwise, proving (8).

Let η be the quadratic character. For any nonprincipal χ, counting the square roots of t and then substituting $t = -s$ gives

$$\begin{aligned} S_\chi := \sum_{u \in \mathbf{F}_p} \chi(u^2+1) &= \sum_{t \in \mathbf{F}_p} (1 + \eta(t))\chi(t+1) \\ &= \eta(-1) J(\eta, \chi), \qquad J(\alpha,\beta) = \sum_{s \in \mathbf{F}_p} \alpha(s)\beta(1-s). \end{aligned} \tag{9}$$

For completeness, for a nonprincipal character α its Gauss sum $G(\alpha) = \sum_t \alpha(t) e^{2\pi i t/p}$ satisfies

$$|G(\alpha)|^2 = \sum_{r \neq 0} \alpha(r) \sum_{y \neq 0} e^{2\pi i (r-1)y/p} = (p-1) - \sum_{r \neq 0,1} \alpha(r) = p.$$

If $\alpha, \beta, \alpha\beta$ are nonprincipal, decomposing the product of Gauss sums by $t + s$ proves $G(\alpha)G(\beta) = J(\alpha,\beta)G(\alpha\beta)$. Hence $|J(\alpha,\beta)| = \sqrt{p}$. If $\beta = \alpha^{-1}$, the substitution $w = s/(1-s)$ instead gives $J(\alpha,\alpha^{-1}) = -\alpha(-1)$, of modulus one. Thus $|S_\chi| \le \sqrt{p}$, including the exceptional case $\chi = \eta$.

Sum (8) over $a = u^2 + 1$. Its principal term is exactly $p\theta_p$, since these values are nonzero. For each nontrivial squarefree $d \mid N$, the $\varphi(d)$ characters cancel the denominator in the triangle inequality. There are $2^{\omega(N)} - 1$ such divisors, which proves (7). □

There are six primes below 16, and $2 \le \ell^{1/4}$ for every prime $\ell \ge 17$. Consequently

$$2^{\omega(N)} \le 64 N^{1/4}. \tag{10}$$

Combining (6)–(10), and using $0 \le \theta_p \le 1$, proves (4). In particular,

$$D_p(x) = 1 - \theta_p + o(1) \quad \text{for every fixed } x \in [1/2, 1), \quad p \to \infty \text{ in } \mathcal{S}. \tag{11}$$

Alternatively the older estimate [3, (5)–(6), Lemma 1] specializes directly with $Q(t) = t^2+1$, $e_1 = 1$, $e_2 = N$. Here $Q(0) = 1$ is not primitive, so its $N(1,N)$ equals P_p; the principal term is $p-1$ because its sum omits $t = 0$. That formulation gives the same $o(1)$ bridge. The stronger exact principal term in the displayed elementary proof is not a new character-sum theorem.

3 Two explicit incompatible subsequences

The arithmetic input for the second subsequence is the classical unconditional mean

$$\frac{1}{\pi(X)} \sum_{p \le X} \theta_p \longrightarrow A := \prod_{\ell \text{ prime}} \left(1 - \frac{1}{\ell(\ell-1)}\right). \tag{12}$$

It is the $r = m = 1$ specialization of the established shifted-prime calculation in Menici–Pehlivan [4, (1), Lemma 2, pp.262–263]; it also follows from Aivazidis–Sofos [5, Theorem 1.1(i), $n = 1$]. It is not the primitive-root conjecture for a fixed base.

Here is a short proof of the needed mean, subject to the standard Siegel–Walfisz theorem. Möbius inversion gives

$$\sum_{p\le X}\theta_p=\sum_{d\le X-1}\frac{\mu(d)}{d}\,\pi(X;1,d).$$

Set $K=(\log X)^6$. Since $\pi(X;1,d)\le X/d$, the tail $d>K$ is $O(X/K)$. Uniformly for $d\le K$, Siegel–Walfisz gives $\pi(X;1,d)=\mathrm{Li}(X)/\varphi(d)+O(X/(\log X)^4)$; the summed error is $O(X\log K/(\log X)^4)=o(X/\log X)$. The coefficient series $\sum_d 1/(d\varphi(d))$ converges: $\varphi(d)\ge\sqrt{d/2}$ follows by checking prime powers in $\varphi(d)^2/d$ (only the factor 2^1 can contribute below one). The resulting absolutely convergent Euler product is A. Finally $\pi(X)\sim\mathrm{Li}(X)$ proves (12). This is a restatement of a classical mean argument.

We need only a coarse positive bound. The factor at 2 is $1/2$, and

$$\sum_{\ell\text{ odd prime}}\frac{1}{\ell(\ell-1)}\le\sum_{\substack{n\ge3\\n\ne4}}\frac{1}{n(n-1)}=\frac{5}{12}.$$

The inequality $\prod(1-a_i)\ge1-\sum a_i$, first for finite products and then by passage to the limit, yields

$$A\ge\frac{7}{24}>\frac{13}{48}.\tag{13}$$

Let Q_j be the product of the first j odd primes. The congruences $p\equiv3\pmod4$ and $p\equiv1\pmod{Q_j}$ specify a reduced residue class modulo $4Q_j$. By the Chinese remainder theorem and Dirichlet's theorem, choose an increasing prime sequence p_j in these classes. Then

$$0\le\theta_{p_j}\le\frac12\prod_{\ell\mid Q_j}(1-1/\ell)\longrightarrow0$$

by Euler's vanishing prime product. Thus (11) gives $D_{p_j}(x)\to1$ for every fixed $x\in[1/2,1)$.

For every odd prime, $\theta_p\le1/2$. If only finitely many $p\in\mathcal{S}$ satisfied $\theta_p\ge1/24$, the prime number theorem in the two odd classes modulo 4 would give

$$\limsup_{X\to\infty}\frac{1}{\pi(X)}\sum_{p\le X}\theta_p\le\frac12\cdot\frac12+\frac12\cdot\frac{1}{24}=\frac{13}{48},$$

contradicting (12)–(13). An increasing sequence $q_j\in\mathcal{S}$ therefore satisfies $\theta_{q_j}\ge1/24$. Equation (11) gives $\limsup_j D_{q_j}(x)\le23/24$, proving (5). The two sequences are independent of x; the exact identity is eventual for each fixed x. The numerical bound is deliberately coarse and is not offered as a new arithmetic refinement.

4 Positive-density selections and weak convergence

The two subsequences alone do not exclude convergence on every further positive-density selection. For that quantifier we use the following older scalar distribution theorem.

Classical scalar input. There is an atomless probability measure ν on $[0,1]$ such that

$$\frac{1}{\pi(X)}\sum_{p\le X}\delta_{\theta_p}\implies\nu\qquad(X\to\infty).\tag{14}$$

Equivalently, the empirical distribution function of θ_p is continuous in the limit. This is the shifted-prime law associated with Kátai. It is recorded directly for this scalar in Deshouillers–Hassani [6, §§1–2, Proposition 2.1 with $m = 2$], and in Aivazidis–Sofos [5, Theorem 1.3(i), $n = 1$, and §6]. In the latter formulation the theorem is stated over prime powers; proper powers number $O(\sqrt{X}\log X) = o(\pi(X))$, so the same empirical law holds over primes, as its proof explicitly explains. Continuity, rather than absolute continuity, is the property used here. No hypothesis such as GRH is imposed on this input.

Lemma 3. *If a bounded real sequence b_p, indexed by primes, has an atomless limiting empirical law, it cannot converge along a prime subset of positive upper density relative to the primes.*

Proof. Suppose $b_p \to c$ on a set $\mathcal{A}$ with upper density $\delta > 0$. For each $\varepsilon > 0$, all but finitely many members of $\mathcal{A}$ have $b_p \in [c-\varepsilon, c+\varepsilon]$. Choose endpoints $X_j \to \infty$ realizing the upper density. The limiting empirical measure assigns the closed interval at least δ: its endpoints have measure zero, so its frequencies converge. Letting $\varepsilon \downarrow 0$ and using continuity from above forces an atom of mass at least δ at c, a contradiction. □

If $D_p(x)$ converged along a set satisfying (3), then (11) would force θ_p to converge there. The density of $\mathcal{S}$ and Lemma 3, applied to (14), exclude that possibility.

To treat weak convergence without a choice of continuity points, use the bounded continuous function on $[0,\infty)$

$$h(t) = \max\{0, \min\{1, 2t-1\}\}.$$

For a proper-divisor period $d \mid p-1$, $d/p < 1/2$ and $h(d/p) = 0$. A maximal period contributes $h((p-1)/p) = 1 - 2/p$. Consequently the exact identity

$$\int h\, d\mu_p = \left(1 - \frac{2}{p}\right)\frac{(p-1)P_p}{p^2} = \theta_p + o(1) \tag{15}$$

holds as $p \to \infty$ in $\mathcal{S}$, by Lemma 2 and (10). Weak convergence along a positive-upper-density subset would force the left side to converge. Lemma 3 again gives a contradiction. This proves part (iii) of Theorem 1. In particular, both conclusions survive deletion of any set of prime density zero. No assertion about arbitrary zero-density selections is required.

5 All-odd-prime robustness under exceptional-fiber conventions

The source permits suitable positive-density primes. The preceding full-affine theorem concerns precisely $\mathcal{S}$; for $p \equiv 1 \pmod 4$ the two multiplier-zero fibers contract to their zero sections and the full affine map is not a permutation. The following additional deduction addresses choosing primes outside $\mathcal{S}$ under natural, explicitly stated point-period conventions. It uses the same credited whole-prime scalar law (14), without a conditional residue-class law.

Proposition 4. *For every odd prime p, put $N = p-1$, $\theta_p = \varphi(N)/N$, $r_p = \#\{u : u^2+1 = 0\} \in \{0,2\}$, and*

$$\Omega_p = \{(u,v) \in \mathbf{F}_p^2 : u^2+1 \neq 0\}, \qquad |\Omega_p| = p(p-r_p).$$

Let P_p now count the nonzero primitive multipliers u^2+1 for this prime. Then, uniformly as $p \to \infty$ through all odd primes,

$$\left|\frac{P_p}{p} - \theta_p\right| \le \frac{r_p\theta_p}{p} + \frac{\theta_p(2^{\omega(N)}-1)}{\sqrt{p}} \le \frac{2}{p} + 64p^{-1/4} = o(1). \tag{16}$$

On Ω_p the map is a permutation. Define its uniform point-period measure μ_p^{reg}, normalized by $p(p-r_p)$ and with periods divided by p. For fixed $1/2 \le x < 1$ and $p(1-x) > 1$,

$$D_p^{\mathrm{reg}}(x) = 1 - \frac{(p-1)P_p}{p(p-r_p)} = 1 - \theta_p + o(1), \tag{17}$$

$$\int h \, d\mu_p^{\mathrm{reg}} = \left(1 - \frac{2}{p}\right) \frac{(p-1)P_p}{p(p-r_p)} = \theta_p + o(1). \tag{18}$$

The same asymptotics hold if any subset of the exceptional fibers is deleted, all regular points are retained, the remaining exceptional points receive arbitrary finite nonnegative assigned periods, and uniform point weights are renormalized. This includes deleting all exceptional fibers, deleting just their nonperiodic points, and completing all p^2 points. Under every such convention, the CDF at each fixed $x \in [1/2, 1)$ and the period measures fail to converge along every prime set of positive upper density relative to all primes.

Proof. The root count is $r_p = 0$ in class 3 and $r_p = 2$ in class 1 modulo 4. The character indicator (8), extended by zero at zero, contributes zero at these roots. Its principal term is therefore exactly $(p - r_p)\theta_p$. The Jacobi-sum calculation (9) applies to every odd prime: $|S_\chi| \le \sqrt{p}$ for each nonprincipal χ, and $S_\eta = -1$ in either residue class. The same squarefree-divisor triangle inequality gives

$$|P_p - (p - r_p)\theta_p| \le \theta_p(2^{\omega(N)} - 1)\sqrt{p}.$$

Division by p, followed by (10), proves (16) uniformly over all odd primes.

There are $p - r_p$ regular fibers. Their zero sections have period one; the other points have the nonzero multiplier's order. The proper-divisor and primitive-period count used in (6) proves the exact CDF formula (17); the globally bounded continuous ramp h from the preceding section proves (18). In both formulas $(p-1)/(p-r_p) = 1 + O(1/p)$, uniformly for $r_p \in \{0, 2\}$, giving the stated asymptotics.

Write $R = p(p - r_p)$. If e exceptional points are retained, $0 \le e \le r_p p$, the resulting probability measure on $[0, \infty)$ is

$$\widetilde{\mu}_p = \frac{R}{R+e}\mu_p^{\mathrm{reg}} + \xi_p, \qquad \xi_p([0,\infty)) = \frac{e}{R+e} \le \frac{r_p}{p}.$$

For every statistic f taking values in $[0, 1]$, its integrals against these two measures differ by at most $r_p/p \le 2/p$; for a bounded real f the bound is $2\|f\|_\infty r_p/p$. This proves the claimed $O(1/p)$ stability of CDF indicators and of h. Assigned periods may exceed p, so no support restriction to $[0, 1]$ is imposed on $\widetilde{\mu}_p$. The ramp is bounded and continuous on the entire nonnegative real axis. In particular its moment remains a legitimate weak-convergence test under arbitrary such assignments.

If a completed/deleted CDF at a fixed $x \in [1/2, 1)$ converged on a positive-upper-prime-density set, (17) and the stability bound would force θ_p to converge there. Weak convergence of the measures would similarly force convergence of the globally bounded ramp moment, hence of θ_p by (18). The credited atomless whole-prime law (14) and Lemma 3 exclude both. □

Only exceptional points and their weighting conventions vary in this proposition; actual cycles on the invariant regular domain are unchanged. It does not assert that the source specifies a unique convention for nonperiodic points, or that primes in class 1 give full-affine permutations. The original class-3 theorem, including its explicit limsup/liminf bound, remains the full-affine statement.

6 Attribution, historical boundary, and reproducibility

The mechanism is a short application of established results. The specific assertion documented here combines the exact point-period identity (6) with classical arithmetic inputs for this fixed map and the statistic in (2). A bounded search of the closest inspected finite-field dynamics sources did not authenticate an earlier statement of this exact application. This does not certify worldwide absence, historical firstness, or present open-problem status.

A specific unresolved bibliographic lead must be retained. Roberts [7, p.220, reference 9] cites N. Neumärker, J. A. G. Roberts, C.-M. Viallet and F. Vivaldi, *The dynamics of reversible birational maps over finite fields*, UNSW Preprint (2011). The body and its version relationships were not recovered in the bounded audit. Citation contexts cannot determine whether that unseen manuscript contains the present application. If an exact prior statement is later authenticated, this note's historical description requires reconciliation; none of the mathematical deductions depend on assuming its absence.

The accompanying source and verification archive contains the TeX source, a dependency-free exact finite checker, source pointers, and byte-preserved original research files with manifests. The new checker passes 147,619 exact finite checks over the 87 admissible primes through 1000, including eight complete affine orbit graphs with 6,088 points and six bad-reduction controls. It verifies fiber orders, divisor fixed-point counts, the point-weighted CDF identity and the continuous moment, and tests the endpoint and bad-reduction controls. A separate all-odd-prime checker verifies the regular-domain formulas and completion/deletion bounds through 1000, including assignments with normalized periods exceeding one. Its exact assertion count and prime range are recorded separately; it does not replace or inflate the original 147,619-check receipt. Each run receipt specifies the actual range and assertion count. Historical AI audit receipts also record 8,347,742 finite assertions and 230,780 analytic controls; they are preserved as earlier runs, not attributed to the new checker. The original SymPy-dependent scripts are retained; this preparation does not claim a successful replay of those scripts in an environment without SymPy. Finite computations check finite identities and implementation consistency. The infinite-prime conclusions follow from the proofs and the explicitly cited classical inputs, not from numerical extrapolation.

AI disclosure and review status. AI systems were extensively involved in the research, mathematical model and proof development, computation, literature investigation, manuscript writing, and adversarial verification. The author is responsible for the submitted claims. This is an unrefereed preprint; it has not undergone formal independent human peer review. Internal AI reviews are not human peer review.

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