A fixed-area exceptional set for positive squared-pole sums

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Abstract

For every finite positive sum $g(z) = \sum_j \lambda_j (z - z_j)^{-2}$ with $\sum_j \lambda_j = 1$, we construct a compact set $S(g)$ of area exactly π such that

$$\int_{\{|z|<R\}\setminus S(g)} |g(z)|\, dA \le 2\pi \log R + \pi(\log 2 + e^{-1}) \qquad (R > 1).$$

The same set serves every radius, with no restriction on the poles. This gives an affirmative deduction for Baernstein's question, Hayman–Lingham Problem 7.37, from the established Eremenko–Hamilton indicator estimate and Levine–Peres quadrature-set construction. The logarithmic coefficient is sharp; no optimality is claimed for the additive constant. Historical first priority is not certified. AI tools were used extensively; this note is unrefereed.

1 The question and the result

Write $D_r = \{z \in \mathbb{C} : |z| < r\}$ and use unnormalized planar Lebesgue area $dA = dx\, dy$. Baernstein's question in [1, Problem 7.37, printed pp. 171–172] asks for an absolute constant C and, for each positive normalized squared-pole sum, an area-π set independent of R, with integral at most $2\pi \log R + C$. Here is an explicit answer.

Theorem 1. *Let $n \ge 1$, $z_j \in \mathbb{C}$, $\lambda_j > 0$, and $\sum_{j=1}^n \lambda_j = 1$, and set $g(z) = \sum_{j=1}^n \lambda_j/(z - z_j)^2$. There exists a compact $S = S(g)$, of area exactly π, such that, simultaneously for every real $R > 1$,*

$$\int_{D_R\setminus S} |g|\, dA \le 2\pi \log R + \pi(\log 2 + e^{-1}). \tag{1}$$

These are ordinary finite nonnegative Lebesgue integrals. Poles may coincide. No smaller logarithmic coefficient works with a finite absolute additive constant.

2 The established inputs

Use the normalization

$$Tf(z) = -\frac{1}{\pi}\,\mathrm{pv}\int_{\mathbb{C}} \frac{f(w)}{(w - z)^2}\, dA(w).$$

For bounded-set indicators this has its usual L^2 realization. Off the closed support the integral converges absolutely. Eremenko–Hamilton [2, printed p. 2794, proof pp. 2796–2797] prove, for measurable $E \subset D_1$,

$$\int_{D_1\setminus E} |T1_E|\, dA \le |E| \log \frac{\pi}{|E|}. \tag{2}$$

The zero-area value is defined by continuity. Their compact capacity-one ambient-set formulation includes the closed unit disk; disk boundaries are null. Measurable indicators follow also from compact approximation, L^2 continuity and Fatou. Scaling $z = L\zeta$ gives

$$\int_{D_L \setminus E} |T1_E| \, dA \le a \log \frac{\pi L^2}{a}, \qquad E \subset D_L, \quad a = |E|. \tag{3}$$

Indeed $T1_E(L\zeta) = T1_{E/L}(\zeta)$. The indicator estimate uses the conformal-on-E case of [2, Theorem 1(i)]. The later correction to its combined general-distortion statement does not change this case [4, Theorem 5 and correction discussion].

The second input is the following consequence of the smash-sum construction in Levine–Peres [3]. Given finitely many points a_j and positive masses m_j, there is a bounded open Ω, containing each disk B_j of area m_j centered at a_j, such that

$$|\partial\Omega| = 0, \qquad |\Omega| = \sum_j m_j, \qquad \int_\Omega h \, dA = \sum_j m_j h(a_j) \tag{4}$$

for every real integrable harmonic function on the whole of Ω. Connectedness is unnecessary. Precise locators in the inspected 1 May 2009 author manuscript are definition (4), Proposition 2.12, Corollary 2.13, Lemma 2.15, Lemma 6.1 and Proposition 6.6 (pp. 6, 12–17, 54–55, 58–59). Proposition 6.6 gives the superharmonic inequality; apply it to h and $-h$.

Here the regularity hypotheses allow finite iteration: at each binary step the sum of two domain indicators is bounded, compactly supported, continuous almost everywhere off null boundaries, and takes values $0, 1, 2$. The density-gap condition of Proposition 2.12 holds with any $0 < \lambda < 1$. That proposition gives a null boundary, Corollary 2.13 gives area additivity, and Lemma 2.15 gives boundedness. Containment, associativity and Proposition 6.6 give (4) for the finite smash sum. The disk mean-value identity supplies the point evaluations. These are established quadrature and partial-balayage results, with antecedents credited in [3]; their existence or regularity is not claimed as new here.

3 One set and its entire exterior field

Combine repeated poles by adding their weights. Apply (4) with $a_j = z_j$ and $m_j = \pi\lambda_j$, and put $S = \overline{\Omega}$. Then S is compact and

$$|S| = |\Omega| = \pi.$$

It contains an open neighborhood of every pole. Fix $z \notin S$. The real and imaginary parts of $w \mapsto (w - z)^{-2}$ are bounded integrable harmonic functions on Ω, so (4) gives

$$\int_\Omega \frac{dA(w)}{(w - z)^2} = \pi \sum_j \frac{\lambda_j}{(z_j - z)^2} = \pi g(z), \qquad T1_\Omega(z) = -g(z). \tag{5}$$

The identity holds on every component of $\mathbb{C} \setminus S$, including any bounded holes. Harmonic tests are on the whole domain, so no analytic continuation from the unbounded component is being assumed. All these integrals are absolutely convergent. Moreover, outside S, $|z - z_j| \ge \sqrt{\lambda_j}$, hence $|g(z)| \le n$. This already proves local finiteness. The set S is fixed before any R is chosen.

4 Localization and the uniform constant

Fix $R > 1$. Set

$$L = \sqrt{2}R, \qquad E = \Omega \cap D_L, \qquad F = \Omega \setminus D_L, \qquad a = |E| \in [0, \pi].$$

By (5), linearity and (3),

$$\int_{D_R \setminus S} |g| \, dA \le a \log \frac{2\pi R^2}{a} + \int_{D_R \setminus S} |T1_F| \, dA, \tag{6}$$

since $D_R \setminus S \subset D_L \setminus E$. For $|w| = \rho > R$, polar integration gives

$$\int_{D_R} \frac{dA(z)}{|z - w|^2} = \int_0^R \frac{2\pi r}{\rho^2 - r^2} \, dr = \pi \log \frac{\rho^2}{\rho^2 - R^2}. \tag{7}$$

The angular identity follows by the geometric series for the Poisson kernel. For $w \in F$, $\rho \ge \sqrt{2}R$, so (7) is at most $\pi \log 2$. The positive separation $L - R$ justifies the ordinary transform integral throughout D_R. Tonelli yields

$$\int_{D_R \setminus S} |T1_F| \, dA \le \frac{1}{\pi} \int_F \int_{D_R} \frac{dA(z) \, dA(w)}{|z - w|^2} \le (\pi - a) \log 2.$$

Consequently,

$$\begin{aligned} \int_{D_R \setminus S} |g| \, dA &\le a \log \frac{2\pi R^2}{a} + (\pi - a) \log 2 \\ &= 2a \log R + \pi \log 2 + a \log \frac{\pi}{a} \\ &\le 2\pi \log R + \pi \log 2 + \frac{\pi}{e}. \end{aligned}$$

The last inequality uses $\log R > 0$, $a \le \pi$ and $\max_{0 \le t \le 1} t \log(1/t) = 1/e$. Continuous endpoint values cover $a = 0$ and $a = \pi$. Only the auxiliary split depends on R. This proves (1) for the same S at every radius.

For sharpness of the leading coefficient, take $g(z) = z^{-2}$. For any measurable S of area π, write $A = D_R \setminus D_1$. The integrand is at least 1 on D_1 and at most 1 on A, while

$$|S \cap A| \le |S \setminus D_1| = |D_1 \setminus S|.$$

Thus the integral gained on $D_1 \setminus S$ dominates the finite integral lost on $S \cap A$, and

$$\int_{D_R \setminus S} |z|^{-2} \, dA \ge \int_A |z|^{-2} \, dA = 2\pi \log R.$$

This also holds when the left side is infinite. Letting $R \to \infty$ excludes a smaller coefficient with any finite additive constant. No optimality of the additive constant in (1) is asserted.

5 Attribution, priority limits and verification

This note supplies the displayed target-specific deduction from established results. It claims no new Beurling estimate, quadrature construction, or general balayage/operator mechanism. Related

recent work includes Lin's partial-balayage principles [5] and Guerra's real-input Beurling weak-type estimates [6]; those inspected operator conclusions do not themselves state the fixed-set, all-radius assertion proved here. The generic ingredients used above were available by 2009. A bounded priority audit identified no verified earlier full-target answer; worldwide first discovery/publication is not certified. In particular, Leung preliminary reports titled *On a fixed area problem*, AMS *Abstracts* 11 (1990), p. 181, and 13 (1992), p. 440, were located through primary indexes, but their mathematical texts were inaccessible. Their relevance remains unresolved. Other unavailable full/final texts and exact inspection scopes are listed in the supporting audit. Neither historical open labels nor repository timestamps establish first priority.

The source archive preserves the original proof, source identities, three distinct mathematical reviews, priority reviews, the correction check and Leung supplement. Supplementary standard-library programs test kernel normalization, quadrature fields (including holes), scaling, scalar bounds and deliberate wrong alternatives. Floating calculations are diagnostics, not validated quadrature or proof certificates; the theorem rests on the universal argument above and its explicitly stated established inputs. AI tools were used extensively in proof development, computation, adversarial checking, literature research and writing. This paper and its supplements are unrefereed and have not undergone external human peer review, journal acceptance or formal proof-assistant certification.

References

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