Five outgoing centers in a 16-realization cube move

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Abstract

We specify six algebraic boundary points, an incoming center, and at least five distinct outgoing centers for a cube move of 16-realizations. Existence follows from strict rational sign bounds for two continuous one-variable residuals. The boundary parameters are uniquely defined real roots of monotone polynomials. This contradicts the at-most-three expectation for construction curves sharing a focus in Remark 4.8 of Melotti–Ramassamy–Thévenin, in the realization setting allowing crossings. The fixed boundary is crossed, so the example concerns no proper embedding. Neither the exact number of centers for this boundary nor a universal maximum is determined.

1 Statement and setting

A quadrilateral $PQRS$ is an α-quad when

$$|P-Q|^\alpha + |R-S|^\alpha = |P-S|^\alpha + |Q-R|^\alpha.$$

We use the realization setting of [1, Definitions 2.1, 2.4–2.5]: crossed faces are permitted. For fixed distinct $A_1, \dots, A_6$, an incoming center Z must make

$$A_1A_2A_3Z, \quad A_3A_4A_5Z, \quad A_5A_6A_1Z$$

α-quads; an outgoing center X must make

$$A_2A_3A_4X, \quad A_4A_5A_6X, \quad A_6A_1A_2X$$

α-quads. Both alternating boundary triples must be noncollinear.

Theorem 1. *There exist six fixed distinct boundary points, with both alternating triples noncollinear, that admit an incoming* 16*-realization and at least five distinct outgoing* 16*-realization centers. Each selected center is distinct from every boundary point.*

The construction also gives at least five intersections of two construction curves with a common focus. Thus it contradicts the $\alpha > 1$ expectation in [1, Remark 4.8, p. 811], and the maximum-three conjecture in the crossing-permitted setting of [2, p. 1785]. The boundary itself has a strict crossing, as verified below. The result makes no claim about proper embeddings, the exact total count, a universal maximum, or $\alpha < 1$.

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2 Five common-focus solutions

Put

$$A=(0,0),\quad B=(b,0),\quad C=(c,d),\qquad b=\frac{2073}{10000},\ c=\frac{4719}{10000},\ d=\frac{64}{10000},\quad \lambda=\frac{27}{1250}. \tag{1}$$

First seek solutions of

$$\|X-B\|^{16}-\|X-A\|^{16}=\lambda,\qquad \|X-C\|^{16}-\|X-A\|^{16}=1. \tag{2}$$

For $s\ge 0$, all roots below denote the nonnegative real eighth root:

$$u=s^{1/8},\quad v=(s+\lambda)^{1/8},\quad w=(s+1)^{1/8},$$

$$x=\frac{b^2+u-v}{2b},\qquad y=\frac{c^2+d^2+u-w-2cx}{2d},\qquad G(s)=x^2+y^2-u. \tag{3}$$

These functions are continuous on $[0,\infty)$.

Lemma 1. *Zeros of G correspond bijectively to solutions of (2), with $s=\|X-A\|^{16}$.*

Proof. The two coordinate equations are the invertible linear system

$$2bx=b^2+u-v,\qquad 2cx+2dy=c^2+d^2+u-w;$$

its determinant is $4bd>0$. If $G(s)=0$, these equations give $|X-A|^2=u$, $|X-B|^2=v$, and $|X-C|^2=w$. Taking eighth powers gives (2). Conversely, a solution determines $s=|X-A|^{16}$; taking the unique nonnegative eighth roots of its three distance powers gives the same linear system and $G(s)=0$. No extra solutions are introduced by squaring or a branch choice. Distinct s give distinct X. □

The accompanying exact verifier establishes the strict bounds in Table 1; every terminating decimal here and below denotes an exact rational.

s	interval containing $G(s)$
9/10000	$(0.38, 0.39)$
11/10000	$(-0.28, -0.27)$
13/10000	$(0.51, 0.52)$
1/2	$(-0.91, -0.90)$
1	$(9, 10)$
10^{16}	$(-4.766, -4.765)$

Table 1: Strict rational sign bounds. The last row also has the analytic derivation below.

The five open intervals between successive cutpoints are disjoint and have opposite endpoint signs. The intermediate value theorem and Lemma 1 therefore give at least five distinct solutions. No uniqueness within any interval is assumed.

For the last row, $u=100$ exactly. Concavity gives, for $a>0$,

$$0<(10^{16}+a)^{1/8}-100\le\frac{a}{8\cdot 10^{14}}.$$

Set $\delta_v = v - u$, $\delta_w = w - u$, and

$$x_0 = \frac{b}{2} = \frac{2073}{20000}, \qquad y_0 = \frac{c^2 + d^2 - bc}{2d} = \frac{1249057}{128000}.$$

Then (3) gives the rational boxes

$$x_0 - \frac{\lambda}{16b\,10^{14}} \le x \le x_0,$$
$$y_0 - \frac{1}{16d\,10^{14}} \le y \le y_0 + \frac{(c/b)\lambda}{16d\,10^{14}}.$$

Both coordinates are positive in these boxes. Squaring their endpoints and subtracting 100 proves the last row, hence in particular $G(10^{16}) < -4.7$. For scale,

$$x_0^2 + y_0^2 = \frac{39007985196121}{409600000000} < 95.235.$$

This argument avoids subtracting nearly equal large radicals. More generally $v - u, w - u \to 0$ as $s \to \infty$, so $G(s) \to -\infty$; the fifth IVT interval nevertheless gives a finite point.

3 One fixed admissible boundary

Let $J(q_x, q_y) = (-q_y, q_x)$. For $q \neq 0$ and the nonzero heights used below, define $T(q, h, \ell)$ as the real solution of

$$|q|^{16}\left(((t-1)^2 + h^2)^8 - (t^2 + h^2)^8\right) = \ell. \tag{4}$$

This is a unique, unambiguous algebraic specification for rational inputs. Indeed, for $g(t) = (t^2 + h^2)^8$,

$$g''(t) = 16(t^2 + h^2)^6(h^2 + 15t^2) > 0.$$

Thus $g(t-1) - g(t)$ is strictly decreasing. Its leading term is $-16t^{15}$, so it tends to $+\infty$ and $-\infty$ at the respective ends of the real line. Existence and uniqueness follow.

Define

$$\begin{gathered} t_1 = T(C, 10, 1), \qquad t_3 = T(B, 1, \lambda), \qquad t_5 = T(C - B, -10, 1 - \lambda), \\ \begin{array}{lll} A_1 = t_1 C + 10JC, & A_2 = A, & A_3 = t_3 B + JB, \\ A_4 = B, & A_5 = B + t_5(C - B) - 10J(C - B), & A_6 = C. \end{array} \end{gathered} \tag{5}$$

Their defining equations give exactly

$$\|A_1 - C\|^{16} - \|A_1 - A\|^{16} = 1, \quad \|A_3 - B\|^{16} - \|A_3 - A\|^{16} = \lambda, \quad \|A_5 - C\|^{16} - \|A_5 - B\|^{16} = 1 - \lambda. \tag{6}$$

In particular, writing $a = \|A_1 - A\|^{16}$, $f = \|A_3 - A\|^{16}$ and $g = \|A_5 - B\|^{16}$, the six successive boundary edge powers are

$$(a,\ f,\ f + \lambda,\ g,\ g + 1 - \lambda,\ a + 1). \tag{7}$$

The odd and even sums both equal $a + f + g + 1$. This exact telescope will supply the dependent third face equations.

Rational bisection of (4), with strict polynomial signs at the endpoints, yields

	x	y
A_1	$(0.17194, 0.17196)$	$(4.72219, 4.72221)$
A_3	$(-0.55229, -0.55227)$	$2073/10000$
A_5	$(0.40359, 0.40361)$	$(-2.64281, -2.64279)$.

Together with (1), these give six distinct boundary points. The exact verifier also gives

$$\det(A_4 - A_2, A_6 - A_2) = 0.00132672 > 0,$$
$$6.37 < \det(A_3 - A_1, A_5 - A_1) < 6.39.$$

Both alternating triples are noncollinear. In fact all twenty boundary triples are noncollinear.

No solution of (2) is a boundary point. Positivity of λ implies $|X-B| > |X-A|$, hence $x < b/2$; this excludes A_1, A_5, whose certified x coordinates exceed $b/2$. The point A fails because $b^{16} \neq \lambda$. The points B, C fail the first and second equations, respectively, by positivity of their right-hand sides. Finally exact intervals give

$$\|A_3 - C\|^{16} - \|A_3 - A\|^{16} - 1 \in (0.98, 0.99),$$

which excludes A_3.

4 An incoming center and the six face equations

Define the fixed algebraic constants

$$L = \|B - A_5\|^{16} - \|B - A_3\|^{16} = g - f - \lambda, \qquad M = \|A - A_1\|^{16} - \|A - A_3\|^{16} = a - f. \tag{8}$$

Exact boundary enclosures prove $L, M > 0$. Put $b' = A_5 - A_3$, $c' = A_1 - A_3$, and $D' = \det(b', c')$. This determinant equals the positive odd determinant above. For $s \in [1/100, 11/1000]$ let

$$u' = s^{1/8}, \qquad v' = (s+L)^{1/8}, \qquad w' = (s+M)^{1/8},$$
$$R_1 = |b'|^2 + u' - v', \qquad R_2 = |c'|^2 + u' - w',$$

$$Y_x = \frac{R_1 c'_y - R_2 b'_y}{2D'}, \qquad Y_y = \frac{b'_x R_2 - c'_x R_1}{2D'}, \qquad H(s) = Y_x^2 + Y_y^2 - u'.$$

The boundary roots and L, M are fixed constants throughout, not values selected anew from interval boxes. All radical arguments are positive and $D' \neq 0$, so H is continuous. Rational interval propagation through these formulas proves

$$H(1/100) \in (0.00037, 0.00038), \qquad H(11/1000) \in (-0.00081, -0.00080).$$

The intermediate value theorem gives a root. The same squared-distance argument as in Lemma 1, translated by A_3, shows that $Z = A_3 + Y$ satisfies

$$\|Z - A_5\|^{16} - \|Z - A_3\|^{16} = L, \qquad \|Z - A_1\|^{16} - \|Z - A_3\|^{16} = M. \tag{9}$$

For the whole retained s interval, interval arithmetic encloses Z in

$$(0.1677, 0.1717) \times (-0.0036, -0.0026),$$

which is disjoint from every boundary point.

For completeness, all six face identities are displayed here. Write $r_i = \|Z - A_i\|^{16}$ and $q_P = \|X - P\|^{16}$ for any of the five selected X. Equations (2), (7), (8) and (9) give

incoming face	opposite-side identity
$A_1A_2A_3Z$	$a + r_3 = r_1 + f$
$A_3A_4A_5Z$	$(f + \lambda) + r_5 = r_3 + g$
$A_5A_6A_1Z$	$(g + 1 - \lambda) + r_1 = r_5 + (a + 1)$

outgoing face	opposite-side identity
$A_2A_3A_4X$	$f + q_B = q_A + (f + \lambda)$
$A_4A_5A_6X$	$g + q_C = q_B + (g + 1 - \lambda)$
$A_6A_1A_2X$	$(a + 1) + q_A = q_C + a.$

The third incoming equation uses exactly $M - L = a - g + \lambda$; it is not inferred from an interval containing zero. This proves Theorem 1 without invoking a flip-existence theorem.

5 Exact verification and scope

The support file `verify_five_centers.py` uses only Python's standard library. All validity decisions use integers and `Fraction`; no floating-point value decides a sign. For $q \geq 0$, it bounds $q^{1/8}$ between consecutive points $k/10^{36}$ and $(k + 1)/10^{36}$, found by binary search and checked by integer eighth powers. Ordinary outward rational interval operations then establish the displayed bounds. The three boundary roots use 128 exact bisections from $[-8, 8]$, with strict endpoint signs. Dependence lost in repeated interval occurrences only enlarges an enclosure; the signs remain strict for the same fixed algebraic boundary.

The file `exact_certificate.json` records the full rational enclosures. A default verifier run recomputes and compares it without writing. Each of five sign brackets is refined by 100 sign-preserving bisections, retaining five disjoint zero brackets without assuming uniqueness. The resulting center boxes and the whole incoming box avoid every line through a boundary pair. Internal check counts are implementation bookkeeping, not counts of independent mathematical propositions.

To see concretely why this is not an embedding example, let $\mathrm{or}(P, Q, R) = \det(Q - P, R - P)$. The exact certificate proves the sign pattern

	$\mathrm{or}(A_1, A_2, A_3)$	$\mathrm{or}(A_1, A_2, A_4)$	$\mathrm{or}(A_3, A_4, A_1)$	$\mathrm{or}(A_3, A_4, A_2)$
sign	$-$	$+$	$+$	$-$

so the fixed boundary segments $[A_1, A_2]$ and $[A_3, A_4]$ cross in their interiors. No central-point choice can make that fixed drawing a proper embedding. Exact total root counts and a global maximum remain undetermined, and no conclusion for $\alpha < 1$ is drawn.

Earlier weighted-distance families and priority. The curves themselves belong to earlier weighted-distance families. Ash–Bolker [3] study costs $\varphi(|x - P|) + w(P)$, including $\varphi(t) = t^4$; their separate oscillatory example $\varphi(t) = t + k \sin t$ is not a pure power and does not itself settle the present count. Bourne–Peletier–Roper [4, Definition 4.12 in the author manuscript] explicitly define p-power cells by comparison of $|x - x_i|^p - w_i$; Bourne–Roper [5, Section 5, p. 2568] repeat this family. With $p = 16$ and weights $w_A = 0$, $w_B = \lambda$, $w_C = 1$, equality of the three costs is exactly (2). No new family of curves is claimed here.

A bounded search of primary texts and public records completed on 15 October 2026 located no earlier pure-power shared-focus counterexample with at least four intersections in the checked corpus. That bounded negative finding does not establish historical priority. No firstness, independence from unknown work, or contemporaneous public-awareness claim is made.

AI assistance and review status. AI systems were used extensively in proof exploration, drafting, code generation and adversarial mathematical checks. This preprint is unrefereed and has not received conventional human peer review.

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References

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