A five-list obstruction on an eleven-vertex tree

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Abstract

We exhibit an eleven-vertex tree T with ordinary $L(2,1)$ initial-segment palette size 5 and arbitrary-list parameter 6. An explicit assignment of five-element positive-integer lists is infeasible by a short forcing argument. An ordinary five-palette labeling and the known $\Delta + 3$ tree list bound give the exact values. This contradicts the universal equality stated in Kohl's 2006 Conjecture 4 and dissertation Conjecture 4.4. Independent pendant-path extensions give infinitely many examples of unbounded diameter. Historical priority, minimal order and present openness are not claimed.

1 Conventions and the historical target

An $L(2,1)$-labeling of a finite simple graph assigns positive integers to vertices so that adjacent labels differ by at least 2, and labels at distance 2 differ. Write $\chi^{(2,1)}(G)$ for the least k allowing such a labeling from the common initial segment $[k] = \{1, \dots, k\}$. Write $\chi_\ell^{(2,1)}(G)$ for the least k such that *every* assignment of k-element subsets $L(v)$ of the positive integers admits a labeling $f(v) \in L(v)$. Lists need not be intervals. Ordinary zero-based span $\lambda(G)$ satisfies $\chi^{(2,1)}(G) = \lambda(G) + 1$.

Kohl's Oberwolfach contribution [1, Conjecture 4, p. 416] and dissertation [2, Conjecture 4.4, p. 124] state $\chi_\ell^{(2,1)}(T) = \chi^{(2,1)}(T)$ for every tree T. The example below is a full negative answer to that stated universal equality. The identification is historical; it does not assert that the conjecture remained open immediately before this work.

2 The obstruction

Let T have the path $a - b - c - d - e$. Attach leaves a_1, a_2 to a, b_1 to b, d_1 to d, and e_1, e_2 to e. These are all vertices and edges, so T has eleven vertices, ten edges, maximum degree 3, radius 3, and diameter 6. Set

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{1, 3, 4, 5, 6\}.$$

Assign B to a_1, a_2, e_1, e_2 and A to every other vertex.

Theorem 1. *The displayed assignment has no $L(2,1)$-labeling, while $\chi^{(2,1)}(T) = 5$, or equivalently $\lambda(T) = 4$, and $\chi_\ell^{(2,1)}(T) = 6$.*

Proof. In a putative list labeling, the three neighbors of b all have list A and are pairwise at distance 2. If $f(b) = 2, 3, 4$, respectively, at most two elements of A differ from $f(b)$ by at least 2. Thus $f(b) \in \{1, 5\}$. The same argument applies to d. Since b, d are at distance 2, they receive different

endpoint labels. Reversing the spine and its attached leaves preserves the assignment, so assume $f(b) = 1$ and $f(d) = 5$. Then $f(c) = 3$.

The edge ab and the distance-two pair ac force $f(a) \in \{4, 5\}$. Each of a_1, a_2 has list B, must differ by at least 2 from $f(a)$, and must avoid $1 = f(b)$. If $f(a) = 4$, only 6 remains; if $f(a) = 5$, only 3 remains. In either case the two leaves receive the same label, although they are at distance 2. This contradiction proves infeasibility.

An ordinary labeling from [5] is

v	a	b	c	d	e	a_1	a_2	b_1	d_1	e_1	e_2
$f(v)$	1	5	3	1	5	3	4	2	4	2	3

as checked directly on edges and pairs of neighbors. A degree-three vertex cannot be labeled from [4]: each possible center label leaves at most two admissible neighbor labels, whereas its three neighbors must be distinct. Hence $\chi^{(2,1)}(T) = 5$. The bad five-lists imply $\chi_\ell^{(2,1)}(T) \geq 6$; Lemma 1 gives the reverse inequality. □

Lemma 1 (Known tree bound)**.** *For every nonempty finite tree S, $\chi_\ell^{(2,1)}(S) \leq \Delta(S) + 3$.*

Proof. Root the tree and label in breadth-first order. For a nonroot vertex v, its parent is its only earlier neighbor, excluding at most three integers from $L(v)$. Earlier vertices at distance 2 are its grandparent, if present, and earlier siblings. There are at most $\Delta(S) - 1$ of these: if the parent is the root there is no grandparent, and otherwise the parent has at most $\Delta(S) - 1$ children. They exclude at most one label each. At most $\Delta(S) + 2$ labels are therefore forbidden, leaving a choice from any $(\Delta(S) + 3)$-element list. The root has no earlier constraints. This also covers a singleton. □

For positive integers $d \geq s$, the more general tree bound $\chi_\ell^{(d,s)}(S) \leq s\Delta(S) + 2d - 1$ is attributed to Voigt in [1, p. 416 and reference 8 on p. 417], citing Kohl, Schreyer, Tuza and Voigt [3]. The proof above is included for self-contained verification, without a novelty claim for the bound.

3 Infinitely many countertrees

Corollary 1. *Attach one path, of any independently chosen nonnegative finite length, outward from each of the six leaves of T. Every resulting tree S has $\chi^{(2,1)}(S) = 5$ and $\chi_\ell^{(2,1)}(S) = 6$. The family has unbounded diameter.*

Proof. The old tree is an induced subtree, its old distances are unchanged, and the maximum degree remains 3. Extend the displayed ordinary labeling successively along each new path. The parent excludes at most three labels of [5], and the grandparent at most one, so a choice always remains. New vertices on different added paths have distance at least 4; no further constraints arise. Thus $\chi^{(2,1)}(S) \leq 5$, with the reverse inequality by restriction to T. Extend the infeasible lists on T by arbitrary five-element lists on new vertices. Any labeling would restrict to a forbidden labeling of T, so $\chi_\ell^{(2,1)}(S) \geq 6$, while the tree bound gives $\chi_\ell^{(2,1)}(S) \leq 6$. □

4 Verification, attribution and limitations

The proof is elementary and independent of computational search. The accompanying verification archive contains the exact graph, lists and ordinary witness, TeX source, a standard-library Python

enumerator, expected results and byte manifests. Its exhaustive finite counts are $0, 8, 6336$ for common palettes [4], [5], [6], respectively, and 0 for the displayed five-lists. These computations check fixed instances; the universal six-list bound and the entire infinite family follow from the proofs, not from sampling. A separate construction-input reconstruction used different constraint propagation and tree-state mechanisms, froze its results before comparison, and did not read the candidate proof. The construction was supplied to it, so this is not an independent discovery claim. Supplementary recurrence details include explicit local-state admissibility; the main obstruction needs no recurrence.

General-graph separations were known earlier. Fiala and Škrekovski's 2001 preprint [4, Fig. 4, p. 14] displays a ten-vertex cyclic graph with an ordinary five-palette labeling and infeasible five-lists. That figure does not itself contradict tree equality. A bounded primary-source and citation audit did not verify an earlier negative tree resolution. Relevant inaccessible or unlocated source bodies remain, as recorded in the supporting priority audit. Accordingly this note makes no claim of historical firstness, minimality, a new obstruction mechanism, or present openness. The inspected 2001 preprint is cited as such; its content is not silently identified with the differently titled 2005 journal version.

AI tools were used extensively in the research, proof development, literature review, computation, verification and writing. The internal independent and adversarial checks were AI-assisted. This preprint is unrefereed: no external human peer review or formal proof-assistant certification is claimed.

References

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[3] A. Kohl, J. Schreyer, Z. Tuza and M. Voigt, List version of $L(d, s)$-labelings, *Theoretical Computer Science* **349**(1) (2005), 92–98. doi:10.1016/j.tcs.2005.09.032.

[4] J. Fiala and R. Škrekovski, *List Distance Labelings of Graphs*, KAM-DIMATIA Series 2001-530; ITI Series 2001-28, 2001. Institutional preprint, Fig. 4, p. 14.