

A sharp forest bound and an improved whisker bound for finite WARM graphs

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21 October 2026 Version 1.0.0 Preprint

Abstract

Consider power reinforcement with exponent $\alpha > 1$ on a finite simple loopless graph: vertices fire at positive rates and select an incident edge proportionally to its current count to the power α . We prove that the infinitely reinforced edges almost surely form a forest for $\alpha > 4/3$, for arbitrary positive vertex rates and positive integer initial counts. The known triangle obstruction shows that the constant $4/3$ cannot be decreased; its equality case is excluded. For equal rates we prove that every surviving component has diameter at most three when $\alpha \geq 17/4$. The main deterministic step excludes an arbitrary weighted odd support cycle of length m when $\alpha > \sec^2(\pi/(2m))$; the corresponding uniform cycle threshold and the even-cycle method are known. We use the published finite WARM convergence and support theorems, and credit the earlier least-weight and champion arguments. The forest result answers the forest-bound improvement requested in Oberwolfach Reports 12/2023; the suggested sharp whisker bound remains open. AI tools were extensively involved. This is an unrefereed preprint.

1 The problem, model and contribution

Let $G = (V, E)$ be a finite simple undirected loopless graph. At each non-isolated vertex v an independent Poisson clock has rate $\lambda_v > 0$. Each edge has an initial count $N_0(e) = n_e \in \{1, 2, \dots\}$. At a firing of v , an incident edge e is selected with probability

$$\frac{N_t(e)^\alpha}{\sum_{f \ni v} N_t(f)^\alpha},$$

and its count is increased by one. Isolated vertices have no effect. Define

$$E_\infty = \{e : N_t(e) \longrightarrow \infty\}, \quad E_+ = \{e : \liminf_{t \rightarrow \infty} N_t(e)/t > 0\}.$$

A whisker is a tree of diameter at most three; an isolated vertex is harmless.

Kleptsyn's contribution to Oberwolfach Reports 12/2023 [1, printed pp. 653–655] asks to improve the sufficient forest bound $\alpha > 2$, allowing unequal rates, and the sufficient equal-rate whisker bound $\alpha > 25$. These bounds come from Holmes–Kleptsyn [3], with the full stochastic bridge supplied by Hirsch–Holmes–Kleptsyn [4]. The report separately suggests that $4/3$ might be sharp for both conclusions. An improvement request and this stronger sharpness suggestion are distinct.

Theorem 1. *For every graph and every positive integer initial count vector as above:*

1. *If $\alpha > 4/3$, then (V, E_∞) is almost surely a forest, for arbitrary positive rates.*
2. *If the rates are equal and $\alpha \geq 17/4$, then almost surely every component of (V, E_∞) is a whisker.*

For each $1 < \alpha < 4/3$ the equal-rate triangle with unit initial counts has positive probability that all three edges survive. Consequently no smaller constant can replace $4/3$ in the first assertion. No assertion is made at $\alpha = 4/3$.

The new mechanism considered here is the weighted odd-cycle estimate, which applies to every equilibrium with that support, even with unequal rates, chords and attachments. The threshold for the *uniform* equilibrium on a bare odd cycle was already given in van der Hofstad–Holmes–Kuznetsov–Ruszel [2, Proposition 2(iv)]. The even-cycle cancellation is already in [4, Lemma 2], whose proof extends immediately to positive rate coefficients. The triangle obstruction is also known [2, Proposition 2(v), Theorem 4(ii)]. The whisker proof sharpens the established least-weight/champion approach. The earlier *stated* bound is 25; in fact the displayed constants in [3, Section 5.1], together with [4], already imply the weaker improvement $\alpha > 24$. A precise derivation is recorded in Section 5. Thus we do not claim the first possible numerical improvement of 25.

2 The stochastic input and initialization

Put $\Lambda = \sum_v \lambda_v$, discarding isolated vertices. If E is empty the conclusions are immediate. Otherwise $\Lambda > 0$. Write

$$S_v(a) = \sum_{e \ni v} a_e^\alpha, \quad F_e(a) = -a_e + \sum_{v \ni e} \lambda_v \frac{a_e^\alpha}{S_v(a)}. \quad (1)$$

We use counts per unit time, so $\sum_e a_e = \Lambda$ at equilibrium. The probability normalization $p_v = \lambda_v/\Lambda$, $x = a/\Lambda$ satisfies $F_\lambda(a) = \Lambda F_p(a/\Lambda)$; the Jacobians are identical.

Theorem 1 of [4, printed p. 3016] proves, from unit initial counts on a finite graph, almost-sure convergence of $(N_t(e)/t)_{e \in E}$ to an equilibrium whose Jacobian has no positive eigenvalue. It also gives positive probability of convergence to each strictly linearly stable equilibrium. Proposition 1 of that paper identifies $E_\infty = E_+$ almost surely under bounded degrees and bounded positive rates; these hypotheses are automatic for our finite graph. These published stochastic results are imported, not reproved here.

Lemma 2. *Every almost-sure tail assertion just described, including the support conclusions deduced from them, transfers from unit counts to any fixed positive integer initial count vector.*

Proof. In the embedded jump chain, prescribe $n_e - 1$ reinforcements of each edge e , in any fixed order. At each step prescribe one endpoint of that edge to fire. The vertex probability is $\lambda_v/\Lambda > 0$, and every prescribed edge probability is positive because all counts are positive. The event that this finite prescription occurs has positive probability, and leaves exactly the count vector n . Conditional on this event, the future chain has precisely the law started at n . The Poisson memoryless property gives the corresponding continuous-time future law. Failure of an almost-sure tail assertion with positive probability from n would therefore give failure with positive probability from unit counts. Finite prefixes and finite time shifts do not change convergence of counts divided by time, their limit, or which counts diverge. This proves the assertion. $\square$

This one-way argument does not transfer positive attainment of every specified stable equilibrium to every initialization. We use attainment only from the unit state for the known triangle obstruction. Arbitrary positive noninteger initial offsets, initially zero edges, loops and infinite graphs are outside the stated model.

At a stochastic limit a , each positive-rate vertex has an incident positive edge: its incident count sum dominates the number of firings at that vertex, so its limiting incident-weight sum is at least λ_v . Thus every $S_v(a) > 0$. Restrict to the positive support H . For $\alpha > 1$, derivatives of a zero coordinate's α th power vanish at zero. Consequently the full Jacobian is the direct sum of the positive-support Jacobian and $-I$ on zero edges. This derivative can equivalently be obtained with the local C^1 extension $x \mapsto (\max\{x, 0\})^\alpha$. Hence a positive eigenvalue on H excludes the full boundary equilibrium. All subsequent calculations are on this positive face.

3 A weighted cycle instability estimate

Define

$$L(a) = -\sum_e a_e + \frac{1}{\alpha} \sum_v \lambda_v \log S_v(a).$$

Then $F = \text{diag}(a)\nabla L$. At an equilibrium, $\nabla L = 0$ and $DF = \text{diag}(a) \text{Hess } L$, which is similar to the symmetric matrix $\text{diag}(\sqrt{a}) \text{Hess } L \text{diag}(\sqrt{a})$. Thus a positive Hessian direction gives a positive real Jacobian eigenvalue, even when other directions are critical.

Use coordinates $y_e = a_e^\alpha$ and write the same potential as

$$\Phi(y) = -\sum_e y_e^{1/\alpha} + \frac{1}{\alpha} \sum_v \lambda_v \log \sum_{e \ni v} y_e.$$

At an equilibrium the Hessians are congruent, since the gradient vanishes. Put

$$b_v = \frac{\lambda_v}{S_v}, \quad c_v = \frac{\lambda_v}{S_v^2}, \quad D_e = a_e^{1-2\alpha}.$$

The equilibrium equation gives

$$a_e^{1-\alpha} = b_u + b_v \quad (e = uv), \tag{2}$$

and direct differentiation gives, for every real edge vector w ,

$$\alpha^2 \text{Hess } \Phi[w, w] = (\alpha - 1) \sum_e D_e w_e^2 - \alpha \sum_v c_v \left(\sum_{e \ni v} w_e \right)^2. \tag{3}$$

No equal-weight or equal-rate assumption occurs here.

Theorem 3. *An equilibrium whose positive support contains an even cycle is linearly unstable for every $\alpha > 1$. If the support contains an odd cycle of length m , it is unstable whenever $\alpha > \sec^2(\pi/(2m))$.*

Proof. Let $e_0, \dots, e_{m-1}$ be a simple support cycle, and let v_i meet e_{i-1}, e_i , with indices modulo m . Put $A_i = a_{e_i}^\alpha$. Equation (2) implies

$$\sum_i D_{e_i} = \sum_i b_{v_i} (A_{i-1}^{-1} + A_i^{-1}).$$

Moreover

$$c_{v_i} = \frac{b_{v_i}}{S_{v_i}} \leq \frac{b_{v_i}}{A_{i-1} + A_i} \leq \frac{b_{v_i}}{4} (A_{i-1}^{-1} + A_i^{-1}),$$

where the last inequality is $(A_{i-1} - A_i)^2 \geq 0$ after clearing positive denominators. Therefore

$$4 \sum_i c_{v_i} \leq \sum_i D_{e_i}. \quad (4)$$

Extra incident positive edges only increase S_v . Set all perturbation coordinates off the cycle to zero, including any chords.

For even m take $w_{e_i} = (-1)^i$. Every incidence sum in (3) is zero and its diagonal term is strictly positive. This is the known alternating mechanism of [4, Lemma 2], with positive vertex coefficients.

For odd m set $\theta = \pi - \pi/m$ and use the two real vectors $u_{e_i} = \cos(i\theta)$ and $w_{e_i} = \sin(i\theta)$. Since $m\theta = (m-1)\pi$ is a multiple of 2π , they are cyclically consistent. At each edge $u_{e_i}^2 + w_{e_i}^2 = 1$, while at each cycle vertex

$$(u_{e_{i-1}} + u_{e_i})^2 + (w_{e_{i-1}} + w_{e_i})^2 = 4 \sin^2(\pi/(2m)).$$

Adding (3) for the two vectors and using (4) yields

$$\alpha^2 (\text{Hess } \Phi[u, u] + \text{Hess } \Phi[w, w]) \geq [\alpha \cos^2(\pi/(2m)) - 1] \sum_i D_{e_i} > 0.$$

At least one vector is a positive Hessian direction, proving instability. $\square$

Every finite nonforest has a simple cycle, and for odd $m \geq 3$ we have $\sec^2(\pi/(2m)) \leq 4/3$. The stochastic input, boundary argument and Lemma 2 prove Theorem 1(1). There is no spurious positive radial eigenvalue: $\sum_e F_e = \Lambda - \sum_e a_e$ implies $\mathbf{1}^T DF = -\mathbf{1}^T$, so positive-eigenvalue vectors lie in the tangent space $\sum_e h_e = 0$.

For the known sharpness obstruction take a triangle with all rates and limiting weights equal to one. Its Jacobian has diagonal $\alpha/2 - 1$ and off-diagonal $-\alpha/4$, giving eigenvalues -1 and $3\alpha/4 - 1$ twice. It is strictly stable for $1 < \alpha < 4/3$, and the published attainment theorem gives positive probability from unit counts. All three edges survive. At $4/3$ the two transverse eigenvalues vanish, and neither this attainment clause nor our strict instability argument decides the stochastic outcome.

4 Equal-rate whiskers

Normalize the common rate to one by time scaling. For $e = uv$ write

$$q = \frac{a_e^\alpha}{S_u}, \quad r = \frac{a_e^\alpha}{S_v}.$$

At equilibrium $a_e = q + r$, where $0 < q, r \leq 1$, hence $a_e \leq 2$. At an equilibrium with no positive eigenvalue, the symmetric Jacobian representative is negative semidefinite. Its diagonal entries give

$$q^2 + r^2 \geq (1 - 1/\alpha)(q + r). \quad (5)$$

For example, direct differentiation gives $J_{ee} = \alpha - 1 - \alpha(q^2 + r^2)/a_e$, proving the condition. It is necessary, not sufficient, for stability.

Lemma 4 (Least-weight argument of [3]). *For $\alpha > 2$, every positive edge at such an equilibrium has weight at least one.*

Proof. Choose an edge of least positive weight in one component. If neither endpoint is a leaf of the support, each has a competing edge of at least the same weight. Then $q, r \leq 1/2$, so $q^2 + r^2 \leq (q + r)/2$, contradicting (5). Thus one endpoint is a leaf and contributes one to the weight. All other weights in the component are no smaller. $\square$

Call an edge internal if neither endpoint is a support leaf. Suppose an internal edge has weight $a \in [1, 3/2]$. Each endpoint has a competitor of weight at least one, so

$$q, r \leq s_\alpha(a) := \frac{a^\alpha}{1 + a^\alpha}, \quad q + r = a. \quad (6)$$

For a feasible split $s \geq a/2$, convexity on $[a - s, s]$ gives

$$\frac{q^2 + r^2}{a} \leq f(a, s_\alpha(a)), \quad f(a, s) = \frac{s^2 + (a - s)^2}{a}. \quad (7)$$

The function f increases in s on $s \geq a/2$, and $s_\alpha(a)$ is nondecreasing in α for $a \geq 1$.

4.1 The closed interval $17/4 \leq \alpha \leq 7$

For $3 \leq B \leq 7$ and $1 \leq a \leq 3/2$, one has

$$\frac{1}{2} \leq \frac{s_B(a)}{a} \leq \frac{2}{3}. \quad (8)$$

Indeed the lower bound is equivalent to $a^{B-1}(2 - a) \geq 1$, which follows from

$$a^2(2 - a) - 1 = (a - 1)(1 + a - a^2) \geq 0.$$

For the upper bound use $s_B \leq s_7$ and $R(a) = 2a^7 - 3a^6 + 2 > 0$. Its derivative is $2a^5(7a - 9)$, so its minimum for $a \geq 0$ occurs at $9/7$, with $R(9/7) = 52763/823543 > 0$.

Put $z = s_B(a)/a$. Direct differentiation and (8) give

$$\frac{d}{da} f(a, s_B(a)) = 1 - 2z^2 + 2(2z - 1)s'_B(a) \geq 1/9 > 0.$$

If $A \leq \alpha \leq B$, it follows that

$$\frac{q^2 + r^2}{a} \leq f(a, s_B(a)) \leq f(3/2, s_B(3/2)). \quad (9)$$

For $T \geq (3/2)^B$, put $S = T/(1+T)$. The table gives exact positive margins $\delta = (1 - 1/A) - f(3/2, S)$:

$[A, B]$	T	S	$f(3/2, S)$	δ
$[17/4, 13/3]$	29/5	29/34	1325/1734	1/1734
$[13/3, 9/2]$	25/4	25/29	3869/5046	163/65598
$[9/2, 5]$	243/32	243/275	117039/151250	5399/1361250
$[5, 6]$	729/64	729/793	991335/1257698	74117/6288490
$[6, 7]$	2187/128	2187/2315	8580639/10718450	527104/16077675

The two nonintegral power caps follow exactly from

$$(29/5)^3 - (3/2)^{13} = \frac{504313}{1024000} > 0, \quad (25/4)^2 - (3/2)^9 = \frac{317}{512} > 0.$$

The other caps are equalities. All $S \geq 3/4$, so the second-argument monotonicity is applicable. The five closed intervals cover $[17/4, 7]$, and (9) is strictly less than $1 - 1/A \leq 1 - 1/\alpha$, contradicting (5). Thus no internal edge has weight at most $3/2$ in this exponent interval.

4.2 The unbounded tail $\alpha \geq 7$

Write $a = 1 + r_0$, with $0 \leq r_0 \leq 1/2$. Since $q, r \leq 1$ and $q + r = a$, convexity gives $q^2 + r^2 \leq 1 + r_0^2$. Condition (5) implies

$$h(r_0) := \alpha r_0^2 - (\alpha - 1)r_0 + 1 \geq 0. \quad (10)$$

For $\alpha \geq 7$, $h(0) > 0$ and $h(1/2) = (6 - \alpha)/4 < 0$. Thus (10) forces r_0 below the smaller root of h . Set $C = 3 - \sqrt{2}$. We have $1 < C < 8/5$, $C/\alpha < 1/2$ and $C^2 + C = 7(C - 1)$, whence

$$h(C/\alpha) = -(C - 1)(1 - 7/\alpha) \leq 0.$$

Therefore $r_0 \leq C/\alpha$ and

$$a^\alpha \leq \exp(\alpha r_0) \leq e^C < 6 \leq \alpha - 1.$$

For completeness the strict bound is elementary: $\sqrt{2} > 7/5$ gives $C < 8/5$. The exponential series at $8/5$ through degree four sums to $9067/1875$; its tail from degree five has successive ratios at most $4/15$, so is at most $4096/34375$. Their sum is $510973/103125 < 6$. By (6) both $q, r < (\alpha - 1)/\alpha$. But (5) says that their weighted mean $(q^2 + r^2)/(q + r)$ is at least $(\alpha - 1)/\alpha$, a contradiction. This proves the same internal-edge exclusion for every $\alpha \geq 7$.

4.3 The matching conclusion

Every internal edge consequently has weight greater than $3/2$. For such an edge $e = uv$, its probability at v is greater than $1/2$, since its other endpoint contributes at most one to $a_e = q + r$. Two internal edges cannot meet at a vertex, where their probabilities would sum to more than one. Hence internal edges form a matching. Every cycle has adjacent internal edges and is excluded directly. A tree of diameter at least four also has adjacent internal edges: take the two middle edges of a four-edge path. Every support component therefore has diameter at most three. This proof of the whisker bound does not depend on the new odd-cycle theorem. Apply the published stochastic input and Lemma 2 to finish Theorem 1(2).

5 Priority, attribution and precise limits

The main comparison is with the arbitrary-rate forest threshold in [3, Theorem 3 and Corollary 1], the equal-rate whisker theorem [3, Theorem 2(a)], the uniform odd-cycle calculation in [2, Proposition 2(iv)], and the even-cycle proof in [4, Lemma 2]. We make no new claim for the stochastic convergence or support-identification theorems, the triangle obstruction, or the least-weight/champion mechanism. A bounded current literature audit recovered no earlier theorem giving the arbitrary-equilibrium weighted odd-cycle estimate or either substantive bound in Theorem 1. This is not a certificate of exhaustive priority or worldwide firstness. Repository creation dates do not certify historical public visibility.

The earlier published argument already has a small numerical improvement of its stated 25. Here is an explicit bridge, to avoid understating it. In [3, Section 5.1, Lemmas 5–7] put $c = 9/4$. For $\alpha > 81/4$, its diagonal/least-weight argument gives

$$a_e \in [1, 1 + c/\alpha) \cup (2 - 2c/\alpha, 2].$$

An edge in the first interval must have a leaf endpoint once $\alpha > c(e^c + 1)$: otherwise its champion probability is at most $e^c/(e^c + 1)$ instead of greater than $1 - c/\alpha$. The exact bound

$$e^{9/4} \leq \sum_{k=0}^{12} \frac{(9/4)^k}{k!} + \frac{(9/4)^{13}/13!}{1 - (9/4)/14} = \frac{191714244173193067}{20206532637491200} < \frac{19}{2}$$

gives $c(e^c + 1) < 189/8 < 24$. For $\alpha > 24$ the second interval has lower endpoint greater than $29/16 > 3/2$, so these edges form a matching and the other edges have leaf endpoints. The published convergence and support bridge then yields whiskers. Thus the purely literal request for some improvement of 25 has this earlier consequence; our contribution is the stronger $17/4$ bound, alongside the forest result.

The interval $4/3 < \alpha < 17/4$ in the suggested sharp equal-rate whisker conclusion is unresolved by this paper. The stochastic equality case $\alpha = 4/3$ is also unresolved here. Unequal-rate whisker conclusions are false in general [3, Example 1]. We assert neither an infinite-graph extension nor results for loops or noninteger initial offsets. The finite simple graph model and all quantifiers appear explicitly above.

6 Reproducibility and research disclosure

The proof, including every rational margin and the unbounded tail, is contained in this paper and does not depend on numerical searches. The supporting package preserves the eight original PR artifacts, separate mathematical and priority audits, exact finite controls and instructions for reproduction in a disposable copy. Historical source wrappers label the work partial by requiring the sharper suggested whisker bound; that label is explicitly superseded here for the literal improvement requests. The added integer-count conditioning argument and prior-method credits are identified, and original inputs remain unchanged. Finite rational checks and rejected deliberately false alternatives corroborate the derivations; hashes establish custody rather than proof. No exploratory finite tree search is used as universal evidence. Third-party primary texts and private operational captures are not redistributed. Standard background results are cited at their interfaces.

AI tools were extensively involved in mathematical research, proof development, symbolic and computational checks, adversarial auditing, literature investigation and manuscript preparation. Independent internal AI reviews are not independent human peer review. This manuscript is a preprint and has not undergone formal independent human peer review, journal acceptance or proof-assistant certification.

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