A finite counterexample to the weighted Yamabe heat trace comparison

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Abstract

Morpurgo's volume-normalized weighted Yamabe heat trace comparison, stated for every dimension at least three, is false in dimension three. We specify one smooth positive conformal weight on the round three-sphere whose scaled heat trace exceeds 13/20 at a specified positive time, whereas the round maximum is less than 321/500. The argument applies classical cylindrical conformal geometry and Dirichlet min–max estimates. A finite cylinder length, explicit smooth caps, and exact rational certificates give quantitative separation.

1 The comparison and the finite result

Let g_0 be the unit round metric on $\mathbb{S}^n$, with positive Laplacian, and let

$$Y_0 = \Delta_{g_0} + \frac{n(n-2)}{4}, \qquad Y_W = W^{-1/2} Y_0 W^{-1/2} \quad \text{on } L^2(dV_{g_0}).$$

For smooth $W > 0$ normalized by $\int_{\mathbb{S}^n} W^{n/2}\, dV_{g_0} = \mathrm{vol}(\mathbb{S}^n, g_0)$, Conjecture 1 of Morpurgo's contribution to [1, printed pp. 421–422, item (16)] asks whether the maximum of $t^{n/2} \operatorname{Tr}(e^{-tY_W})$ over $t > 0$ is at most the round maximum.

Theorem 1. *There is a smooth strictly positive W on $\mathbb{S}^3$, with $\int W^{3/2}\, dV_{g_0} = 2\pi^2$, and an explicitly defined $t_* > 0$, such that*

$$t_*^{3/2} \operatorname{Tr}(e^{-t_* Y_W}) > \frac{1106714561}{1700054750} > \frac{13}{20}, \qquad \max_{t>0} t^{3/2} \operatorname{Tr}(e^{-tY_0}) < \frac{321}{500}.$$

Both comparison maxima are attained at positive finite times.

This refutes the universal assertion over $n \geq 3$. The comparison restricted to $n \geq 4$, and the separate monotonicity conjecture in the following source paragraph, remain unaddressed. The weaker original thresholds 13/20 and 129/200 also follow.

Long cylindrical conformal blowups are classical; Ammann and Jammes used Pinocchio metrics for conformally covariant eigenvalue problems [2], with earlier constructions cited there. Their construction, a chosen fixed rescaled cap, ordinary Dirichlet trial modes, and a strict round maximum bound already give a qualitative counterexample. We record a specified finite quantitative application to Morpurgo's comparison. The geometry and spectral mechanism are credited as classical. Related transformed-trace inequalities occur in [3]; their allowed functional classes do not directly state this time-maximum comparison. The accompanying dated priority audit records inspected versions and retrieval limits. Historical priority of the exact application, worldwide firstness, and current openness are unestablished.

*OpenAI tools were used extensively in developing, drafting, coding, and adversarially checking this work. This is an unrefereed preprint; no completed external human peer review or formal proof-assistant verification is claimed.

2 A smooth capped cylinder in the round conformal class

Identify the twice-punctured three-sphere with $\mathbb{R}\times\mathbb{S}^2$, using stereographic radius $r=e^s$. Then

$$g_0=\operatorname{sech}^2(s)(ds^2+g_{\mathbb{S}^2}).$$

Fix $L=10000$. Define

$$\eta(x)=\begin{cases}0 & x\le 0,\\ e^{-1/x} & x>0,\end{cases}\qquad \chi(x)=\frac{\eta(x)}{\eta(x)+\eta(1-x)},\qquad q(s)=|s|-L/2,\qquad u(s)=\exp[-\chi(q(s))\log\cosh(q(s))].$$

The denominator is everywhere positive. The smooth step χ is zero for $x\le 0$, one for $x\ge 1$, and has flat endpoint jets. Thus

$$g_L=u(s)^2(ds^2+g_{\mathbb{S}^2}),\qquad W_L=u(s)^2\cosh^2(s)$$

is the exact unit-radius product cylinder on $|s|\le L/2$. The apparent corner of $|s|$ is harmless because $u=1$ near zero. Flatness gives smooth matching at both ends of each transition. For $|s|\ge L/2+1$, $u(s)=\operatorname{sech}(|s|-L/2)$.

At the positive pole use $\rho=e^{-s}$, and at the negative pole use $\rho=e^s$. In either smooth stereographic chart,

$$W_L=e^L\frac{(1+\rho^2)^2}{(1+e^L\rho^2)^2},\qquad g_L=\frac{4e^L}{(1+e^L\rho^2)^2}(d\rho^2+\rho^2 g_{\mathbb{S}^2}).$$

These coefficients are smooth positive functions of the squared Cartesian radius, including $\rho=0$. Hence $g_L=W_Lg_0$ and W_L extend smoothly to the standard compact sphere. The finite constant e^{10000} requires no floating-point evaluation.

Writing $V_L=\operatorname{vol}(g_L)$, the spherical area is 4π, so

$$4\pi L\le V_L=4\pi\int_{\mathbb{R}}u(s)^3\,ds<4\pi(L+4).\tag{1}$$

Indeed each cap has radial integral at most $1+\int_1^\infty 8e^{-3q}\,dq=1+8/(3e^3)<2$. Set

$$a=(2\pi^2/V_L)^{2/3},\qquad h=ag_L,\qquad W=aW_L,\qquad t_*=4a.\tag{2}$$

These are explicit finite expressions, with V_L the specified integral. They give $\int W^{3/2}\,dV_{g_0}=a^{3/2}V_L=2\pi^2$.

3 The operator and the variational heat lower bound

In dimension three the geometric Yamabe operator is $Y_g=\Delta_g+R_g/8$. Conformal covariance for $h=Wg_0$ and its volume measure give

$$Y_h=W^{-5/4}Y_0W^{1/4},\qquad dV_h=W^{3/2}dV_{g_0}.$$

The unitary map $Uf=W^{3/4}f$ from $L^2(dV_h)$ to $L^2(dV_{g_0})$ therefore satisfies $UY_hU^{-1}=Y_W$. These identities extend from smooth functions to closed elliptic forms: all multipliers and their inverses are smooth on the compact sphere. Moreover

$$\langle Y_Wf,f\rangle=\int\left(|\nabla(W^{-1/2}f)|^2+\tfrac{3}{4}|W^{-1/2}f|^2\right)dV_{g_0}\ge\frac{3}{4\max W}\|f\|^2.$$

Thus the spectrum is strictly positive and every positive-time heat trace is finite. No sign condition on the transition scalar curvature is needed.

On the exact cylinder $R=2$, hence $Y_{g_L}=-\partial_s^2+\Delta_{\mathbb{S}^2}+1/4$. Take the $\mathbb{S}^2$-constant Dirichlet functions $\phi_k(s)=\sin(\pi k(s+L/2)/L)$, extended by zero off the cylinder. They lie in the global H^1 form domain because their boundary traces vanish. A jump in normal derivative contributes no delta to the weak first derivative. Their form and norm matrices are diagonal; on the first k modes the largest Rayleigh quotient

is $\mu_k = 1/4 + (\pi k/L)^2$. Min–max for the full closed-manifold spectrum, including multiplicities, gives $\lambda_k(g_L) \le \mu_k$. Consequently

$$\operatorname{Tr}(e^{-4Y_{g_L}}) \ge e^{-1} \sum_{k\ge 1} e^{-4\pi^2 k^2/L^2} \ge e^{-1}\left(\frac{L}{4\sqrt{\pi}} - 1\right). \tag{3}$$

For the last inequality, a decreasing Gaussian has its integer sum at least its integral from one to infinity; the omitted integral on $[0,1]$ is at most one. Finite sums and monotone limits justify the trace comparison.

Constant rescaling gives $Y_h = a^{-1}Y_{g_L}$. Using (2), including its time and prefactor, yields

$$t_*^{3/2}\operatorname{Tr}(e^{-t_* Y_W}) = \frac{16\pi^2}{V_L}\operatorname{Tr}(e^{-4Y_{g_L}}) > \frac{\sqrt{\pi}}{e}\frac{L-4\sqrt{\pi}}{L+4}.$$

The numerator is positive. The elementary enclosures $1.772 < \sqrt{\pi} < 1.773$ and $e < 2.719$ give

$$t_*^{3/2}\operatorname{Tr}(e^{-t_* Y_W}) > \frac{1772}{2719}\frac{10000000-7092}{10004000} = \frac{1106714561}{1700054750} > \frac{13}{20}. \tag{4}$$

The exact margin in the last comparison is $3357947/3400109500 > 0$. This proves a finite-length estimate.

4 A rational all-time round bound

Round degree $j-1$ harmonics have Laplace eigenvalue j^2-1 and multiplicity j^2. Thus the round scaled trace is

$$F_0(t) = t^{3/2}\sum_{j\ge 1} j^2 e^{-t(j^2-1/4)}.$$

Differentiating Gaussian Poisson summation, justified by locally uniform Gaussian convergence for positive times, gives

$$F_0(t) = \frac{\sqrt{\pi}}{4}e^{t/4}\left[1 + 2\sum_{m\ge 1}(1-2\pi^2 m^2/t)e^{-\pi^2 m^2/t}\right].$$

For $0 < t \le 7/5$ all correction terms are negative, so $F_0(t) < (1.773/4)e^{7/20}$. Write $F_0 = A + B$, where $A(t) = t^{3/2}e^{-3t/4}$. The function A increases up to two; every summand of B decreases for $t \ge 2/5$. For $a \ge 7/5$, the tail beginning at $j = 3$ has consecutive ratio at most $(16/9)e^{-7a} < 1/2$. Hence

$$B(a) \le D(a) := a^{3/2}\left(4e^{-15a/4} + 18e^{-35a/4}\right).$$

For $7/5 \le a \le t \le b \le 2$, $F_0(t) \le A(b) + D(a)$. The following exact-rational upper enclosures cover that range and small time.

time region	upper enclosure
$(0, 7/5]$	$629002/10^6$
$[7/5, 3/2]$	$631338/10^6$
$[3/2, 8/5]$	$636143/10^6$
$[8/5, 17/10]$	$639464/10^6$
$[17/10, 9/5]$	$641172/10^6$
$[9/5, 19/10]$	$641202/10^6$
$[19/10, 2]$	$639541/10^6$

(5)

Here is a complete arithmetic specification for these enclosures. Let $P_N(x) = \sum_{k=0}^{N} x^k/k!$. For $0 \le x < N+2$,

$$P_N(x) \le e^x \le P_N(x) + \frac{x^{N+1}}{(N+1)!}\frac{1}{1-x/(N+2)}.$$

Use its degree-14 upper bound for $e^{7/20}$ and e. Replace each negative exponential in A, D by $1/P_{50}(x)$, and each square root of an endpoint by $(\lfloor 10^8\sqrt{x}\rfloor + 1)/10^8$, computed by integer bisection. The six bounds are

then rational expressions; exact comparison gives (5). The supplied `check_trace_bounds.py` records the endpoint formulas and their reduced fractional values. For the constant enclosures it uses $\pi = 8\arctan(1/3) + 4\arctan(1/7)$, with 24 alternating terms and the next-term remainder for each arctangent. Tangent addition and first-quadrant signs prove the identity. Rational squaring verifies the stated square-root brackets. No floating-point acceptance is used.

For $t \geq 2$ every summand $t^{3/2}e^{-\lambda t}$ is nonincreasing, since $\lambda \geq 3/4$. Summing pointwise inequalities gives $F_0(t) \leq F_0(2)$, already controlled by the last row. Thus

$$\sup_{t>0} F_0(t) \leq 641202/10^6 < 321/500 < 129/200.$$

Together with (4), this proves the comparison in Theorem 1.

5 Maxima, verification, and attribution

For either fixed normalized smooth metric, the leading heat coefficient gives the zero-time limit

$$(4\pi)^{-3/2}2\pi^2 = \sqrt{\pi}/4.$$

At infinity the strictly positive first eigenvalue gives

$$\operatorname{Tr}(e^{-tY}) \leq e^{-(t-1)\lambda_1} \operatorname{Tr}(e^{-Y}), \qquad t \geq 1.$$

The scaled trace tends to zero. Continuity for positive time and (4) imply attainment of the weighted supremum inside a compact positive-time interval. The first round mode at two is $2\sqrt{2}e^{-3/2} > \sqrt{\pi}/4$, so the same argument applies to the round supremum. Its inequality follows, for example, by squaring and using $128 > \pi e^3$ with the certified constant bounds.

The supporting package preserves the original finite proof and three frozen evidence archives, and includes independent geometric, spectral, and bounded-priority audits. The new coarse rational certificate strengthens the original round bound; original certificates remain unchanged. The portable checks establish arithmetic and artifact integrity. Smooth extension, conformal covariance, form domains, min–max, Poisson summation, and heat asymptotics are mathematical proof obligations checked separately by manual adversarial AI review. Standard imported results are used with the hypotheses above; no formal proof-assistant certification is supplied.

The qualitative negative answer is a straightforward application of classical cylindrical geometry and standard spectral estimates. Our finite formulas and certificates make that application checkable for the exact historical question. The source audit found no earlier explicitly assembled exact refutation in its stated corpus and records its limits, including unavailable final or older source versions. This bounded finding does not certify historical priority or current openness.

References

[1] C. Morpurgo, Cwikel–Lieb–Rozenblum bounds and heat kernel inequalities, item (16) in *Problems from the Oberwolfach Workshop, Low Eigenvalues of Laplace and Schrödinger Operators*, Oberwolfach Reports **6** (2009), 355–428, printed pp. 421–422. doi:10.4171/OWR/2009/06.

[2] B. Ammann and P. Jammes, The supremum of first eigenvalues of conformally covariant operators in a conformal class, in *Variational Problems in Differential Geometry*, London Mathematical Society Lecture Note Series **394** (2011), 1–23. doi:10.1017/CBO9780511863219.002. First preprint: arXiv:0708.0529 (2007); inspected author manuscript dated December 8, 2008.

[3] C. Morpurgo, Sharp inequalities for functional integrals and traces of conformally invariant operators, *Duke Math. J.* **114** (2002), 477–553. doi:10.1215/S0012-7094-02-11433-1. The accompanying priority audit identifies the inspected author manuscript and final-version retrieval limits.