Zero Filling and Persistent Free Subgroups of the Figure-Eight Knot

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Abstract

The figure-eight knot gives a negative answer to the universal rank-two free-subgroup question printed in Motegi's 2020 Oberwolfach contribution. It is hyperbolic and has no finite nonmeridional surgery, but no rank-two free subgroup of its knot group injects into every rational filling group. Indeed, zero filling is a torus bundle, so its metabelian fundamental group kills a nontrivial second-derived word in every such free subgroup. We give the short obstruction and verify the no-finite-surgery hypothesis using the classical filling classification. The ingredients are classical; this note records their application and the question's quantifiers. It does not resolve the search for a positive hyperbolic example. Historical priority cannot be fully cleared, and no firstness claim is made.

1 The question and the conclusion

For a knot $K \subset S^3$, let $G = \pi_1(S^3 \setminus \nu K)$, with meridian μ and preferred longitude λ. For a rational slope $r = p/q$, where $\gcd(p, q) = 1$ and $q \neq 0$, write

$$N_r = \langle\langle \mu^p \lambda^q \rangle\rangle_G = \ker(q_r), \qquad q_r : G \longrightarrow \pi_1(K(r)).$$

Thus 0 is included; the meridian $\infty = 1/0$ is excluded. The final question on printed p. 493 of [1] asks whether a hyperbolic knot with no finite nonmeridional surgery admits $H \cong F_2$ such that

$$H \cap \bigcup_{r \in \mathbb{Q}} N_r = \{1\}. \tag{1}$$

Here "finite surgery" means a filling with finite fundamental group. Condition (1) requires that *each* $q_r|_H$ be injective. Injectivity of a map into a product of filling groups would be a weaker, different condition.

Theorem 1. *The figure-eight knot* 4_1 *is hyperbolic and every rational filling* $4_1(r)$ *has infinite fundamental group. Nevertheless, for every subgroup* $H \leq \pi_1(S^3 \setminus \nu 4_1)$ *isomorphic to* F_2*,*

$$H \cap N_0 \neq \{1\}.$$

Consequently (1) *fails for this knot.*

This refutes the affirmative assertion for *every* knot satisfying the stated hypotheses. It says nothing against the existence of some other hyperbolic knot for which (1) holds. Nor does it settle a variant that is allowed to discard finitely many slopes.

2 The algebraic obstruction

Use $[u, v] = uvu^{-1}v^{-1}$ and write $J' = [J, J]$ and $J'' = [J', J']$. A group M is metabelian if $M'' = 1$.

Lemma 1. *If $f : J \to M$ is a homomorphism into a metabelian group, its restriction to every free rank-two subgroup of J has nontrivial kernel.*

Proof. Let a, b be a free basis of such a subgroup H. Put $c = [a, b]$, $d = aca^{-1}$ and $w = [c, d] \in H''$. In the free alphabet $x = a$, $y = b$, $X = a^{-1}$, $Y = b^{-1}$, the freely reduced form of w is

$$\texttt{xyXYxxyXYXyxxYXX}.$$

It is a nonempty reduced word of length 16, so $w \neq 1$ in H, and therefore in J. But $f(w) \in M'' = 1$. □

If a closed three-manifold T is a torus bundle over S^1, the homotopy sequence gives

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \pi_1(T) \longrightarrow \mathbb{Z} \longrightarrow 1. \tag{2}$$

The left injection uses $\pi_2(S^1) = 0$; surjectivity on the right uses the connected fiber. Since the quotient is abelian, $\pi_1(T)'$ lies in the abelian fiber subgroup. Hence $\pi_1(T)$ is metabelian. No monodromy matrix, linearity theorem, or claim about a particular embedding of H is needed. Applying Lemma 1 to zero filling will prove the obstruction once its classical topology and the surgery hypothesis are established.

3 The required figure-eight topology

The figure-eight exterior is hyperbolic; see the ideal-tetrahedron construction in Thurston's notes [2, Chapter 4]. Brittenham–Wu [3, Theorem 1.1(4)] record the complete rational exceptional-slope list, attributing this case to Thurston:

$$E = \{0, \pm 1, \pm 2, \pm 3, \pm 4\}.$$

Every rational slope outside E gives a closed hyperbolic manifold and therefore infinite fundamental group. Indeed, a finite fundamental group would make its universal cover a finite-sheeted cover of a compact manifold, whereas that cover is the noncompact space $\mathbb{H}^3$.

For completeness, all nine remaining slopes also have infinite groups:

Slopes	Classical description	Reason for infinitude
$0, \pm 4$	toroidal	an incompressible torus injects $\mathbb{Z}^2$
± 1	Seifert base $S^2(2, 3, 7)$	hyperbolic triangle quotient
± 2	Seifert base $S^2(2, 4, 5)$	hyperbolic triangle quotient
± 3	Seifert base $S^2(3, 3, 4)$	hyperbolic triangle quotient

The toroidal classification is in [3]. The positive-slope Seifert data are listed in [5, Section 9.4, p. 71]. The three orbifold Euler characteristics are, respectively,

$$-1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{7} = -\tfrac{1}{42}, \quad -1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{5} = -\tfrac{1}{20}, \quad -1 + \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{4} = -\tfrac{1}{12}.$$

A Seifert group surjects onto its base orbifold group. Each of these triangle groups is infinite, for it has the hyperbolic triangle action. Amphichirality gives the negative-slope descriptions independently [2, Chapter 4]. In particular we do not use the displayed 2 versus -3 equality in the inspected version of [5, Section 9.4].

Finally, zero surgery is a torus bundle, as stated explicitly in Khoi's Section 3, unnumbered figure-eight example on printed p. 524 [4]. This also accords with Thurston's torus mapping-torus discussion [2, p. 70]; the latter's D-shaped surgery subscript is not our sole basis for identifying slope zero. Equation (2) now shows that $\pi_1(4_1(0))$ is metabelian, and Lemma 1 supplies a nonidentity element of $H \cap \ker q_0$ for every $H \cong F_2$. This proves Theorem 1. The no-finite hypothesis concerns all rational slopes; the meridional S^3 filling is excluded throughout. Even imposing the absence of reducing surgeries does not remove this example, by [3, Lemma 2.1].

4 Context, credit, and limits

The later question in [6, Section 7.1] is again phrased for an arbitrary hyperbolic knot. Its surrounding discussion motivates the search for a positive hyperbolic example, which Theorem 1 leaves unresolved. We do not infer an unstated intended quantifier or a zero-slope exception.

The topological classification and the group law above are classical. Boden–Friedl [7, Section 4.6, proof of Proposition 24, equation (5)] explicitly relate the genus-one free fiber, its longitude commutator, and metabelian representations. A 2023 precursor [8, p. 37] explicitly records the figure-eight zero filling's abelian commutator subgroup. That passage is cited as historical ingredient evidence only: the precursor's Claim 6.3 was subsequently reported as having a gap, and no proof here relies on that claim. The present proof instead uses the classical topology above.

Two independent primary-source priority audits found no explicit earlier statement of this full counterexample in the text they inspected. This is not a certificate of priority. In particular, the full body of the 2024 cyclic-persistence paper [9] and of the 2019 vanishing-elements paper [10] was not inspected. Some longer precursor versions were scanned and selectively read; Bart's 2001 surgery paper was compared only in relevant publicly indexed text, not in full. Unindexed statements or remarks may already contain this elementary obstruction. Priority cannot be fully cleared; no claim of historical firstness, new topological ingredients, or continued open status in 2026 is made.

The accompanying authored-only verification archive preserves the original inputs, independent derivations and reviews, source locators and fingerprints, and an offline replay driver for exact finite controls. These computations supplement the proof; they neither exhaust arbitrary free subgroups nor replace the cited topology. A pinned exact-inventory gate repairs a historical release verifier's unanchored manifest and constant source-exclusion flag; the original files remain unchanged. Raw third-party papers and account information are excluded. Checksums establish consistency, not authenticity or priority.

AI tools were used extensively in solving, research, writing, reproduction, and adversarial verification. This is an unrefereed preprint with internal AI adversarial reviews. It has not received human peer review or proof-assistant certification.

References

[1] K. Motegi (with T. Ito and M. Teragaito), Dehn fillings on knot groups, in *Low-dimensional Topology*, Oberwolfach Reports 8/2020, 491–493. doi:10.4171/OWR/2020/8.

[2] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, 1980 lecture notes, electronic version 1.1 (2002), Chapter 4. `https://library.slmath.org/nonmsri/gt3m/PDF/4.pdf`.

[3] M. Brittenham and Y.-Q. Wu, The classification of exceptional Dehn surgeries on 2-bridge knots, *Comm. Anal. Geom.* 9 (2001), 97–113. doi:10.4310/cag.2001.v9.n1.a4. The inspected author preprint is headed 1995.

[4] V. T. Khoi, On the integral of $\log x\, dy/y - \log y\, dx/x$ over the A-polynomial curves, *Acta Math. Vietnam.* 33 (2008), 519–528. https://math.ac.vn/public/uploads/files/0803519.pdf.

[5] S. Gukov and C. Manolescu, A two-variable series for knot complements, author version, Section 9.4. https://web.stanford.edu/~cm5/surgeries.pdf.

[6] T. Ito, K. Motegi, and M. Teragaito, Dehn filling and the knot group II: Ubiquity of persistent elements, arXiv:2604.01697v1 (2026). https://arxiv.org/abs/2604.01697v1.

[7] H. U. Boden and S. Friedl, Metabelian $\mathrm{SL}(n,\mathbb{C})$ representations of knot groups, III: deformations, *Q. J. Math.* 65 (2014), 817–840. doi:10.1093/qmath/hat047. Inspected version: https://arxiv.org/abs/1208.1708v1 (2012).

[8] T. Ito, K. Motegi, and M. Teragaito, Dehn filling trivialization on a knot group: separation and realization, arXiv:2303.15738v1 (2023). https://arxiv.org/abs/2303.15738v1. The v2 arXiv record's Comments field reports the Claim 6.3 gap.

[9] T. Ito, K. Motegi, and M. Teragaito, Dehn filling and the knot group III: cyclic persistent subgroups, *Bol. Soc. Mat. Mex.* 30 (2024), 99. doi:10.1007/s40590-024-00674-9. Full body not inspected.

[10] K. Ichihara, K. Motegi, and M. Teragaito, Vanishing nontrivial elements in a knot group by Dehn fillings, *Topology Appl.* 264 (2019), 223–232. doi:10.1016/j.topol.2019.06.023. Full body not inspected.