# The least-period criterion for outer focal products in elliptic billiards

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**Abstract**

Fix a noncircular ellipse and a proper nested confocal elliptical caustic supporting periodic billiards of least period $N \geq 3$. Intersect the boundary tangents at consecutive impact points. The product of the ordinary distances from these outer vertices to either original focus is independent of phase if and only if $4 \mid N$. The two constant products then agree. The positive direction is the primitive interpretation of the invariant $k_{115}$ and is also proved in Ferudun's contemporaneous note, which we explicitly credit. A paired-distance divisor proves both directions, including odd periods and the four-period degree drop. Repeated traversals cannot restore constancy. Historical priority and independent discovery are not certified.

## 1 Statement and provenance

Write

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad C : \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1, \qquad a > b > 0,$$

where

$$a^2 - \alpha^2 = b^2 - \beta^2 = \lambda, \quad 0 < \lambda < b^2, \qquad c^2 = a^2 - b^2 = \alpha^2 - \beta^2. \tag{1}$$

The foci used throughout are $F_\pm = (\pm c, 0)$ of $E$ and $C$. For a billiard with successive impacts $P_j$ on $E$, let $Q_j$ be the intersection of the tangents to $E$ at $P_j, P_{j+1}$. Assume that the fixed pair supports a periodic family of *least* period $N \geq 3$. For either focus put

$$R_\pm = \prod_{j=0}^{N-1} |Q_j - F_\pm|.$$

Distances are ordinary positive Euclidean lengths. Phase varies within the family for this fixed pair and winding.

**Theorem 1.** *Under* (1)*, each of* $R_+$ *and* $R_-$ *is independent of phase if and only if* $4 \mid N$*. When* $4 \mid N$*, their constants are equal. Every admissible coprime star winding is included.*

Reznik, Garcia and Koiller list the positive assertion as $k_{115}$ in Table 2, p. 5 of their 2020 preprint [1], and Table 2, p. 345 of the distinct 2021 journal paper [2]. Those editions give a proof question mark. This is dated provenance, not evidence of present worldwide openness. Their table does not expressly settle a repeated-traversal convention. Theorem 1 uses least period; an unrestricted interpretation based only on the length of a repeated list is false, as Section 4 explains.

Ferudun [3], Theorem 1, proves the same positive assertion with an explicit constant, for the same primitive scope and all coprime windings. His Jacobi factorization and cyclic-divisor argument closely overlap the mechanism below. We present an attributed separate treatment and the necessary-and-sufficient criterion, without claiming firstness or historical novelty of that criterion. The dated records and limits of priority are stated in Section 5.

## 2 Geometry and the paired distance

All Jacobi functions below have modulus $k$, not parameter $k$. Put

$$k = \frac{c}{\alpha} \in (0,1), \qquad k' = \frac{\beta}{\alpha} = \sqrt{1-k^2}, \qquad K = K(k), \quad K' = K(k').$$

Stachel's canonical parametrization [4, Theorem 4.3 and (4.9)] gives $v \in (0, K)$ such that

$$P(u) = (-a \operatorname{sn} u, b \operatorname{cn} u), \qquad a = \frac{\alpha \operatorname{dn} v}{\operatorname{cn} v}, \quad b = \frac{\beta}{\operatorname{cn} v}, \qquad \delta = 2v. \tag{2}$$

The billiard map is $u \mapsto u + \delta$. After choosing orientation, least period is equivalent to

$$\delta = \frac{4K\tau}{N}, \qquad \gcd(\tau, N) = 1, \qquad 0 < \tau < N/2. \tag{3}$$

The real parametrization has period $4K$; reversal only permutes the distance factors. For example, the chord through $P(u-v), P(u+v)$ is

$$-\frac{x \operatorname{sn} u}{\alpha} + \frac{y \operatorname{cn} u}{\beta} = 1,$$

the tangent to $C$ at $(-\alpha \operatorname{sn} u, \beta \operatorname{cn} u)$. The billiard dynamics input in (2) is established background, rather than a new parametrization.

**Lemma 1.** *The boundary tangents at $P(u-v), P(u+v)$ meet at*

$$Q(u) = (-A \operatorname{sn} u, B \operatorname{cn} u), \qquad A = \frac{a \operatorname{dn} v}{\operatorname{cn} v} = \frac{a^2}{\alpha}, \quad B = \frac{b}{\operatorname{cn} v} = \frac{b^2}{\beta}. \tag{4}$$

*For real $u$ this point is finite and outside $E$.*

*Proof.* Put $s = \operatorname{sn} u$, $z = \operatorname{cn} u$, $d = \operatorname{dn} u$, $h = \operatorname{sn} v$, $C_v^0 = \operatorname{cn} v$, $D_v^0 = \operatorname{dn} v$, and $D_0 = 1 - k^2 s^2 h^2$. The tangents have equations $-X \operatorname{sn}(u \pm v)/a + Y \operatorname{cn}(u \pm v)/b = 1$. The Jacobi addition formulas [5, Section 22.8] reduce their sum and difference to

$$-\frac{X s C_v^0 D_v^0}{a} + \frac{Y z C_v^0}{b} = D_0, \qquad -\frac{X z}{a} - \frac{Y s D_v^0}{b} = 0.$$

Their solution is (4). The original determinant, up to sign, is $2hC_v^0 d/(abD_0) > 0$. Finally

$$\frac{Q_x^2}{a^2} + \frac{Q_y^2}{b^2} - 1 = \frac{h^2 \operatorname{dn}^2 u}{\operatorname{cn}^2 v} > 0.$$

Thus neither focus is an outer vertex. □

The actual outer sequence is $Q(u_0+v+j\delta)$; its fixed offset is absorbed into phase. In particular $Q(u+2K)=-Q(u)$. The conic locus $X^2/A^2+Y^2/B^2=1$ is also the polar reciprocal of $C$ in $E$: its equation is $\alpha^2X^2/a^4+\beta^2Y^2/b^4=1$. This construction is classical polarity [6, Section 4]. The locus need not be confocal with $E$; we never replace $F_\pm$ by its own foci.

Define a meromorphic function without choosing complex square roots:

$$H(u)=((Q_x-c)^2+Q_y^2)((Q_x+c)^2+Q_y^2). \tag{5}$$

For real $u$, $H(u)=(|Q(u)-F_+|\,|Q(u)-F_-|)^2>0$.

**Lemma 2.** *For a positive constant $C_v$,*

$$H(u)=C_v\,\mathrm{dn}^2\,u\,[\mathrm{dn}^2\,\delta-k^2\,\mathrm{cn}^2\,\delta\,\mathrm{sn}^2\,u]. \tag{6}$$

*This identity holds before imposing any parity condition on $N$.*

*Proof.* Put $\kappa=k^2$, $t=\mathrm{sn}^2\,v$, $q=1-t$, and

$$U=1-2\kappa t+\kappa t^2,\quad V=1-2t+\kappa t^2,\quad W=1-\kappa t^2.$$

Equation (4) gives

$$A=\frac{\alpha(1-\kappa t)}{q},\quad B=\frac{\alpha k'}{q},\quad A^2-B^2=\frac{\alpha^2\kappa V}{q^2},\quad B^2+c^2=\frac{\alpha^2U}{q^2}.$$

For $S=\mathrm{sn}^2\,u$, expand (5) and use $U+V=2q(1-\kappa t)$ to obtain

$$\begin{aligned}H(u)&=\frac{\alpha^4}{q^4}[(U+\kappa VS)^2-4\kappa(1-\kappa t)^2q^2S]\\&=\frac{\alpha^4}{q^4}(1-\kappa S)(U^2-\kappa V^2S).\end{aligned}$$

Now $\mathrm{cn}\,\delta=V/W$ and $\mathrm{dn}\,\delta=U/W$ by the double-angle formulas. This proves (6) with $C_v=\alpha^4W^2/q^4>0$. The paired factorization is also obtained by multiplying Ferudun's two one-focus factors [3, (5)]. □

## 3 The divisor and the exact period criterion

Work on the compact connected torus

$$T=\mathbb{C}/(2K\mathbb{Z}+2iK'\mathbb{Z}),\qquad p=iK'.$$

The functions $\mathrm{sn}^2\,u$ and $\mathrm{dn}^2\,u$ are meromorphic on $T$. The first has one double pole at $p$; the second has that pole and a double zero at $p+K$. We use the standard shift

$$\mathrm{sn}(z+K+iK')=\frac{\mathrm{dn}\,z}{k\,\mathrm{cn}\,z} \tag{7}$$

and the periods in [5, Tables 22.4.1–3]. In particular $\mathrm{dn}(u+2K)=\mathrm{dn}\,u$.

If $\delta\neq K$, the bracket in (6) has one double pole at $p$ and zeros at $p+K\pm\delta$, by (7). These zeros are distinct modulo the lattice: their coincidence would force $2\delta\in2K\mathbb{Z}$, hence $\delta=K$ since

$0 < \delta < 2K$. Neither is $p$ or $p + K$ in this case. The two distinct zeros exhaust its degree-two zero divisor, and so are simple. Consequently

$$\operatorname{div} H = 2[p + K] + [p + K + \delta] + [p + K - \delta] - 4[p]. \tag{8}$$

There is exactly one exceptional primitive case. If $\delta = K$, (3) and coprimality force $N = 4$, $\tau = 1$; the bracket is the nonzero constant $k'^2$, and instead

$$\operatorname{div} H = 2[p + K] - 2[p]. \tag{9}$$

This degree drop is essential.

Let $G = \langle \delta \rangle \subset T$. Its order is

$$L = \frac{N}{\gcd(N, 2)},$$

because $\delta/(2K) = 2\tau/N$. Define

$$\mathcal{F}(u) = \prod_{j=0}^{L-1} H(u + j\delta), \qquad O_x = \sum_{g \in G} [x + g].$$

For $N \neq 4$, summing translated copies of (8) gives

$$\operatorname{div} \mathcal{F} = 4O_{p+K} - 4O_p. \tag{10}$$

All multiplicities and collisions between different factors are included: translation by $\delta$ or $-\delta$ permutes $G$. The two cosets in (10) coincide if $K \in G$ and are otherwise disjoint. On the real period circle, $K$ is the unique point of order two, so

$$K \in G \iff L \text{ is even} \iff 4 \mid N. \tag{11}$$

Thus $\mathcal{F}$ is constant for $4 \mid N$: its zero divisor makes it holomorphic on the compact torus. At $N = 4$, (9) yields the same cancellation with coefficient two. Its constant is finite and positive, since every factor is positive for real $u$. For the other parities, (10) is a nonzero divisor, hence $\mathcal{F}$ is nonconstant.

*Proof of Theorem 1.* Write $R_\pm(u) = \prod_{j=0}^{N-1} |Q(u + j\delta) - F_\pm|$. For all $N$, the antipodal *phase shift* gives

$$R_-(u) = R_+(u + 2K), \qquad R_+(u)^2 R_-(u)^2 = \mathcal{F}(u)^{N/L}. \tag{12}$$

If either focal product were constant, both would be constant by the first identity. The second would make the meromorphic function $\mathcal{F}^{N/L}$ constant on a real interval. It has no real poles; the identity theorem then makes it constant on $T$. For $4 \nmid N$ this contradicts its nonzero divisor (10). This proves necessity also for odd $N$, without assuming that an antipodal point belongs to the same odd primitive orbit.

If $4 \mid N$, then $L = N/2$ and coprimality makes $\tau$ odd. Now $L\delta = 2K\tau$ really does pair the vertices inside the orbit: $Q(u + (j + L)\delta) = -Q(u + j\delta)$. Therefore

$$R_+(u) = R_-(u), \qquad R_+(u)^2 = \mathcal{F}(u).$$

The unique positive square root of the positive constant proves sufficiency and equality of the two constants. No complex square-root branch is used. □

The use of compact-torus pole cancellation is classical; compare Akopyan, Schwartz and Tabachnikov [7]. The period condition depends on the subgroup, not the chosen coprime generator, so includes every star winding in (3).

## 4 Repeated traversals and an exact boundary example

**Corollary 1.** *For any positive integer $r$, the product on $r$ traversals is phase independent exactly when the primitive product is.*

*Proof.* The repeated product is $R_{\pm}(u)^r$ and $x \mapsto x^r$ is injective on $(0, \infty)$. □

For an explicit same-caustic check, take $a = 3$, $b = 1$ and $\lambda = 9/16$. Two primitive six-periodic orbits have vertices

$$(3, 0),\ (9/4, \sqrt{7}/4),\ (-9/4, \sqrt{7}/4) \quad \text{and their negatives,}$$

and

$$(0, 1),\ (-3\sqrt{15}/4, 1/4),\ (-3\sqrt{15}/4, -1/4) \quad \text{and their negatives,}$$

respectively, in the indicated order followed by the negatives in order. The caustic has squared axes $135/16, 7/16$. Boundary membership, tangency and the reflection law can be checked exactly; the verification package supplies these checks. The first three outer vertices are, respectively,

$$(3, 1/\sqrt{7}),\ (0, 4/\sqrt{7}),\ (-3, 1/\sqrt{7}),$$

and

$$(-9/\sqrt{15}, 1),\ (-12/\sqrt{15}, 0),\ (-9/\sqrt{15}, -1),$$

followed by their negatives. The products to $F_+ = (\sqrt{8}, 0)$ are

$$\frac{20736}{343} \qquad \text{and} \qquad \frac{6912}{125},$$

which are unequal. Traversing each twice produces nominal length twelve and unequal squared products. Ferudun [3, Section 5] gives a different repeated-triangle example for the same interpretive issue; this distinction is explicitly credited.

For a circle, both foci coincide with the center and the outer radius of a regular star is $a/\cos(\pi\tau/N)$, so every admitted period has a constant product. Noncircularity is therefore essential to necessity. Hyperbolic or collapsed caustics, separatrices and $N = 2$ are outside Theorem 1; no conclusion about them is drawn by a limit.

## 5 Attribution, timeline and verification

The submitted positive-direction proof is retained at GitHub PR 200, whose server creation record is 1 October 2026, 05:03:51 UTC, with submitted head `379a95960995`. Ferudun's deposited record [3] reports creation on 2 October 2026, 01:41:04.506303 UTC; DataCite records DOI registration at 01:41:05 UTC that day. These particular witnesses document contemporaneous separate treatments. They do not exclude an earlier website or other repository disclosure, establish absolute first publication, or certify independent discovery. The converse criterion was added and audited on 16 October, and is not backdated to the original submission. A bounded classical and modern literature audit did not establish its historical novelty.

Canonical parametrization, polarity, Jacobi identities and the norm principle are credited background. Older alternating focal-product relations, such as [8, Section 2, p. 219], compare the two foci for a fixed circumscribed polygon; that assertion alone is not phase constancy of the full product. The supplementary priority ledger records exact source scopes and access/search limits. No exhaustive absence-of-prior-work claim is made.

The supplementary verification includes exact coefficient identities, cyclic divisor arithmetic, physical reflection/tangency fixtures and high-precision direct tangent-intersection controls. Numerical samples are diagnostics, not the universal proof. Legacy assertion-based verifiers and their optimized-mode limitation are distinguished from the package's explicit-guard checker. Dependency versions and replay instructions are supplied. No formal proof-assistant certification is claimed.

AI tools were used extensively in developing the original argument, extending and checking it, searching sources, drafting the manuscript and conducting adversarial audits. These audits are AI-assisted checks, not conventional independent human peer review. This is an unrefereed research note; the author is responsible for its claims and presentation.

## References


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