# An exact six-period correction to the printed elliptic billiard assertion `k108`

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**Abstract**

The experimental entry `k108` in Table 2 of Reznik–Garcia–Koiller asserts constancy of the outer-tangent area ratio divided by the product of original half-angle sines for $N \equiv 2 \pmod 4$. Two strictly convex primitive six-period trajectories on one confocal billiard family give 11664/3125 and 3645/1024, disproving the printed assertion. The horizontal six-period coordinates and caustic are prior work of Garcia–Reznik, printed in their November 2020 preprint. This note is an attributed corrective application of established geometry, with exact reproducibility certificates; it claims neither new six-period geometry nor historical firstness.

**Disclosure.** AI tools were used extensively in mathematical development, source comparison, drafting, exact computation, and adversarial checking. This is an unrefereed preprint draft without conventional human peer review. AI checks are fallible and are not proof-assistant formalization. Historical firstness remains unconfirmed. The written proof and supporting computations are provided for independent scrutiny.

## 1 The precise experimental claim and prior work

Table 2 of *Eighty New Invariants of N-Periodics in the Elliptic Billiard*, arXiv:2004.12497v11, 29 October 2020, p. 5 [1], and the version of record *Fifty New Invariants of N-Periodics in the Elliptic Billiard*, p. 345 [2], print

$$k_{103} = A'/A, \qquad k_{105} = \prod_{i=1}^{N} \sin(\theta_i/2), \qquad k_{108} = k_{103}/k_{105} \quad (N \equiv 2 \pmod 4). \tag{1}$$

The proof column for $k_{108}$ is a question mark: this is an unproved experimental constancy assertion, not a theorem. Here $A$ is the original polygon's signed area; $A'$ is the signed area of the polygon formed by consecutive tangents to the billiard boundary at its vertices. It is not the caustic-contact polygon. Section 3.1 calls $\theta_i$ the original polygon's angle, without the adjective "internal." We use ordinary convex internal angles, consistently calibrated by the same table's $k_{101} = \sum_i \cos\theta_i = JL - N$ (checked below).

**Prior construction and scope.** Garcia–Reznik already give the horizontal simple six-period coordinates, their confocal caustic, and perimeter in arXiv:2011.06640v1, 12 November 2020, Appendix B.5.1, p. 20 [3]; the final paper retains them in B.5.1, p. 71 [4]. Our horizontal member is exactly their specialization $a = 2, b = 1$. Their source assumes $a > b > 0$; we verify the vertical

member directly rather than quoting an unlicensed interchange of major and minor axes. Connected family motion and even-period central symmetry are also established results: see Stachel's Theorem 4.3 [6] and Corollary 4.2(i) [5]. The contribution here is the explicit exact evaluation that resolves the particular printed quotient. No new construction, general billiard method, independent-discovery claim, or official erratum is asserted.

## 2 Two physical members of one connected family

Use the fixed pair

$$E : \frac{x^2}{4} + y^2 = 1, \qquad C : \frac{x^2}{32/9} + \frac{y^2}{5/9} = 1. \tag{2}$$

Both squared semiaxes decrease by $\lambda = 4/9 \in (0,1)$, with common focal square 3. Thus $C$ is a nondegenerate confocal ellipse strictly inside $E$. In counterclockwise order take

$$H = \big((2,0),(4/3,\sqrt{5}/3),(-4/3,\sqrt{5}/3),(-2,0),(-4/3,-\sqrt{5}/3),(4/3,-\sqrt{5}/3)\big),$$
$$V = \big((0,1),(-4\sqrt{2}/3,1/3),(-4\sqrt{2}/3,-1/3),(0,-1),(4\sqrt{2}/3,-1/3),(4\sqrt{2}/3,1/3)\big).$$

Each list consists of six distinct points satisfying the equation of $E$, in strictly increasing boundary order. Strict convexity of $E$ therefore gives a simple strictly convex hexagon. The lengths are respectively

$$(1, 8/3, 1, 1, 8/3, 1), \qquad (2, 2/3, 2, 2, 2/3, 2),$$

so both perimeters are $L = 28/3$. Six distinct successive vertices exclude every smaller billiard period.

**Finite contacts.** For a line $\alpha x + \beta y = h$, the support equation for tangency to $C$ is

$$(32/9)\alpha^2 + (5/9)\beta^2 = h^2, \qquad c = ((32/9)\alpha/h, (5/9)\beta/h).$$

The first side of $H$ has line $x + 2y/\sqrt{5} = 2$, contact $(16/9, \sqrt{5}/9)$, and segment parameter $1/3$; its horizontal side has contact $(0, \sqrt{5}/3)$ and parameter $1/2$. The first side of $V$ has line $-x/(2\sqrt{2}) + y = 1$, contact $(-8\sqrt{2}/9, 5/9)$, and parameter $2/3$; its vertical side has contact $(-4\sqrt{2}/3, 0)$ and parameter $1/2$. Coordinate symmetries give all sides, with contact-parameter lists

$$H : (1/3, 1/2, 2/3, 1/3, 1/2, 2/3), \quad V : (2/3, 1/2, 1/3, 2/3, 1/2, 1/3).$$

Every contact is strictly inside its finite side. The center, hence the caustic, is to the left of each directed side.

**Physical reflection.** At $p = (x, y)$ put $n = (x/4, y)$. For incoming and outgoing unit velocities $u, w$, the law is

$$w = u - 2\frac{u \cdot n}{n \cdot n} n. \tag{3}$$

The two vertex types per orbit are as follows; the remaining ones follow by coordinate symmetries, which preserve both ellipses and reflection.

| $p$ | $u$ | $w$ | $n$ |
|---|---|---|---|
| $(2,0)$ | $(2/3, \sqrt{5}/3)$ | $(-2/3, \sqrt{5}/3)$ | $(1/2, 0)$ |
| $(4/3, \sqrt{5}/3)$ | $(-2/3, \sqrt{5}/3)$ | $(-1, 0)$ | $(1/3, \sqrt{5}/3)$ |
| $(0,1)$ | $(-2\sqrt{2}/3, 1/3)$ | $(-2\sqrt{2}/3, -1/3)$ | $(0,1)$ |
| $(-4\sqrt{2}/3, 1/3)$ | $(-2\sqrt{2}/3, -1/3)$ | $(0,-1)$ | $(-\sqrt{2}/3, 1/3)$ |

Substitution gives (3) and $u \cdot n = 1/3 > 0$, $w \cdot n = -1/3 < 0$ at every vertex. Thus these are physical billiard trajectories, with incoming Joachimsthal constant $J = 1/3$, not merely polygons of tangent supporting lines.

**One connected oriented family.** The fixed confocal elliptic pair satisfies the hypotheses of Stachel's canonical parametrization [6, Theorem 4.3]: positive nested semiaxes $a_e = 2, b_e = 1$, $a_c = \sqrt{32}/3, b_c = \sqrt{5}/3$, and $a_e^2 - a_c^2 = b_e^2 - b_c^2 = 4/9$. Its modulus is $\sqrt{27/32} \in (0,1)$. The boundary is parametrized on a circle of length $4K$ by $(-2\,\mathrm{sn}\,u, \mathrm{cn}\,u)$, and the billiard map on a fixed oriented caustic branch is a constant shift. The verified $H$ closes after six steps with turning number one, so that shift is $4K/6$ on the counterclockwise branch. All initial parameters therefore generate primitive six-period trajectories on one connected parameter circle. The first directed chords of $H$ and $V$ both put $C$ on the left, so they select that same branch. Varying the initial boundary point from $(2,0)$ to $(0,1)$ gives a continuous path between them. This uses the established porism/shift theorem, independently of the proposed invariant. Their displayed antipodal pairing also agrees with the known even-period, odd-turn central symmetry [5, Corollary 4.2(i)].

## 3 Exact area and angle evaluation

The tangent at $p \in E$ is $(p_x/4)x + p_y y = 1$. Solving adjacent tangent pairs produces

$$\begin{aligned} Q_H = \big((2,\sqrt{5}/5),(0,3\sqrt{5}/5),(-2,\sqrt{5}/5),\\ (-2,-\sqrt{5}/5),(0,-3\sqrt{5}/5),(2,-\sqrt{5}/5)\big),\\ Q_V = \big((-\sqrt{2},1),(-3\sqrt{2}/2,0),(-\sqrt{2},-1),(\sqrt{2},-1),(3\sqrt{2}/2,0),(\sqrt{2},1)\big). \end{aligned}$$

The adjacent tangent determinants are positive: they are $\sqrt{5}/6$ or $2\sqrt{5}/9$ for $H$, and $\sqrt{2}/3$ or $2\sqrt{2}/9$ for $V$. Thus no intersection is infinite. The displayed $Q$ lists are strictly convex and counterclockwise. The ordinary signed shoelace formula,

$$A(P) = \frac{1}{2}\sum_{i=1}^{6} \det(p_i, p_{i+1}), \qquad p_7 = p_1,$$

gives positive values, so unsigned areas agree as well:

| | $A$ | $A'$ | $R = A'/A$ |
|---|---|---|---|
| $H$ | $20\sqrt{5}/9$ | $16\sqrt{5}/5$ | $36/25$ |
| $V$ | $32\sqrt{2}/9$ | $5\sqrt{2}$ | $45/32$ |

For internal angles, $\cos\theta = -u \cdot w$ and $\sin^2(\theta/2) = (1-\cos\theta)/2$. The two axial vertices of $H$ have cosine $-1/9$, and its other four have $-2/3$; for $V$ the corresponding cosines are $-7/9$ and $-1/3$. All angles belong to $(0,\pi)$, fixing the positive half-sine roots. Antipodal pairs therefore give

$$S_H = \prod_i \sin(\theta_i/2) = \frac{5}{9}\left(\frac{5}{6}\right)^2 = \frac{125}{324}, \qquad S_V = \frac{8}{9}\left(\frac{2}{3}\right)^2 = \frac{32}{81}.$$

Both cosine sums are $-26/9 = JL - 6$, confirming the source's original-angle normalization.

**Proposition 1.** *The printed experimental assertion* (1) *fails at primitive* $N = 6$ *on the family* (2).

*Proof.* The physical, convex, primitive, and common-family conditions were established above, and $6 \equiv 2 \pmod 4$. Direct division gives

$$\mathcal{K}(H) = \frac{36/25}{125/324} = \frac{11664}{3125}, \qquad \mathcal{K}(V) = \frac{45/32}{32/81} = \frac{3645}{1024},$$

with $\mathcal{K}(H) - \mathcal{K}(V) = 553311/3200000 > 0$. □

As two-member consistency controls, $AA' = 320/9$ and $RS = 5/9$ for both members. The half-angle products $S_H, S_V$ are unequal. The common value of $RS$ proves neither a repaired identity at every phase nor an all-period replacement, and no intended author typo is inferred.

## 4 Reproducibility, historical limits, and disclosure

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## References


[1] D. Reznik, R. Garcia, and J. Koiller, *Eighty New Invariants of N-Periodics in the Elliptic Billiard*, arXiv:2004.12497v11 (29 October 2020), Section 3.1 and Table 2, p. 5. https://arxiv.org/abs/2004.12497v11.

[2] D. Reznik, R. Garcia, and J. Koiller, *Fifty New Invariants of N-Periodics in the Elliptic Billiard*, Arnold Math. J. **7** (2021), 341–355, Table 2, p. 345. https://doi.org/10.1007/s40598-021-00174-y.

[3] R. Garcia and D. Reznik, *Invariants of Self-Intersected and Inversive N-Periodics in the Elliptic Billiard*, arXiv:2011.06640v1 (12 November 2020), Appendix B.5.1, p. 20. https://arxiv.org/abs/2011.06640v1.

[4] R. Garcia and D. Reznik, *Exploring self-intersected N-periodics in the elliptic billiard*, Ann. Math. Inform. **55** (2022), 49–75, Appendix B.5.1, p. 71. https://doi.org/10.33039/ami.2022.02.001.

[5] H. Stachel, *The geometry of billiards in ellipses and their Poncelet grids*, J. Geom. **112** (2021), article 40, Corollary 4.2(i). https://doi.org/10.1007/s00022-021-00606-2.

[6] H. Stachel, *On the motion of billiards in ellipses*, Eur. J. Math. **8** (2022), 1602–1622, Theorem 4.3. https://doi.org/10.1007/s40879-021-00524-2.