A missing multiplicity condition for double-permutation sequences

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Abstract

The four term-sequence axioms in the inspected Goodman–Pollack manuscript and the 2008 Oberwolfach problem session do not suffice for the proposed encoding by connected sets with compatible tangent and separating pseudolines. An explicit two-body word has least period eight and satisfies all four axioms, but has no separating switch and hence cannot encode the required internal tangents. This gives a negative answer to the literal universal realization question under those axioms. It leaves intact the representation theorem for valid fibrations and the stronger allowable-interval definition. We also prove that requiring a fully separated term forces the missing switch multiplicities, clarifying the boundary of a separate normalized-input construction request.

Disclosure. AI tools were used extensively in formulating, reproducing, checking and adversarially reviewing the argument, investigating sources, and preparing this note and its software. This is an unrefereed preprint. Internal AI-assisted review is not external human peer review or proof-assistant certification.

1 The exact formulation

Fix an integer $n \geq 2$. A double permutation is an ordering of

$$1, 1', \ldots, n, n'$$

in which i precedes i' for every i. Let $\mathcal{R}$ reverse an ordering and interchange every primed and unprimed index. Set $N = 8\binom{n}{2}$. Consider a doubly infinite word $(P_k)_{k\in\mathbb{Z}}$ satisfying:

1. Each P_k is a double permutation.
2. Each transition $P_k \longrightarrow P_{k+1}$ swaps two adjacent indices belonging to different bodies.
3. $P_{k+N/2} = \mathcal{R}(P_k)$ for every k.
4. The word has period N.

We call these the *four term axioms*. The example below has *least* period N, so it also satisfies the interpretation of the last axiom as a primitive-period requirement.

These are Corollary 11 and Definition 12 of the 26 November 2006 Goodman–Pollack author manuscript [1, pp. 8–9]; the same four conditions are printed in Pollack's problem-session item

in [2, p. 2551]. The topological realization question appears explicitly in the manuscript and as Question 1 in [2, p. 2492]. Our input definition is exactly the four conditions above.

The source geometric encoding uses pairwise disjoint compact connected sets and a compatible arrangement of pseudolines, with a strict separator and four designated common tangents for each body pair: two internal and two external [1, Theorem 1 and Definitions 4, 8–9]. Both directions of every tangent are recorded. A directed tangent is indexed according to the left or right supporting side of each body. Thus an internal tangent, with the bodies on opposite sides, gives an ordered switch involving one primed and one unprimed index. We call such a switch *mixed*. Supporting switches have two primed or two unprimed indices. We use the post-switch ordering convention of [1, p. 7].

Theorem 1. *The four term axioms do not imply realizability in this connected-set, compatible-pseudoline encoding. A counterexample already exists for $n = 2$, with least period eight.*

2 The eight-term obstruction

Put

$$A = (1, 2, 1', 2'), \qquad B = (1, 2, 2', 1'), \qquad C = (2, 1, 2', 1').$$

Repeat the following period in both directions:

$$\boxed{A,\ B,\ A,\ B,\ C,\ B,\ C,\ B.} \tag{1}$$

Every displayed ordering has 1 before $1'$ and 2 before $2'$. Each of the transitions $A \leftrightarrow B$ and $B \leftrightarrow C$ is one adjacent different-body swap. These are all the transitions of (1), including its wrap.

Moreover $\mathcal{R}(A) = C$, $\mathcal{R}(C) = A$ and $\mathcal{R}(B) = B$, so the half-period identity holds at every phase. The least period divides eight. A proper divisor would be 1, 2 or 4 and would imply $P_0 = P_4$. Since $A \neq C$, the least period is eight.

The ordered post-switch pairs, including the last-to-first transition, are

$$\begin{gathered}(2', 1'),\ (1', 2'),\ (2', 1'),\ (2, 1),\\ (1, 2),\ (2, 1),\ (1, 2),\ (1', 2').\end{gathered} \tag{2}$$

Each of the four supporting types occurs twice. None of $(1, 2')$, $(2, 1')$, $(1', 2)$ or $(2', 1)$ occurs.

In the source encoding the two internal tangent pseudolines of this body pair must contribute mixed events. In fact recording both directions gives four such events. Equation (2) has none, so it cannot be that encoding. This proves Theorem 1.

This obstruction uses the prescribed supporting sides and tangent roles, rather than Euclidean projection or simultaneous straightening. The source's strict separators and compatibility hypotheses ensure the unambiguous directed-tangent encoding; the argument requires neither smoothness nor positive area. A degeneracy that does not supply the prescribed tangent roles and single-swap events does not realize (1). Changing phase, reversing traversal, renaming the bodies or switching between before- and after-swap conventions preserves the absence of mixed events.

3 A fully separated input term changes the problem

The last paragraph of the separate polygonal-construction request [2, p. 2551] refers to the fully separated ordering $(1, 1', 2, 2', \ldots, n, n')$. A variant that explicitly requires this ordering in its input excludes (1). More generally:

Lemma 2. *If a word satisfying the four term axioms contains*

$$D = (i_1, i'_1, i_2, i'_2, \ldots, i_n, i'_n)$$

for some ordering $i_1, \ldots, i_n$ of the body labels, then every ordered different-body endpoint pair occurs exactly once as a switch in a full period.

Proof. Shift the phase to make $P_0 = D$. Its half-period counterpart $\mathcal{R}(D)$ reverses the order of the body blocks and preserves each body's internal order. The relative order of each of the four endpoint pairs belonging to any two different bodies must therefore reverse. There are exactly $4\binom{n}{2} = N/2$ such pairs.

An adjacent transposition changes the relative order of exactly the two endpoints it swaps. Each required endpoint pair must consequently be swapped an odd, positive number of times between D and $\mathcal{R}(D)$. There are precisely $N/2$ steps available, so every required pair is swapped once and none can backtrack. Reverse-toggle transports the second half to the first, giving one swap of each unordered different-body endpoint pair in each half. Since the full period returns every relative order to its starting value, the two swaps of each such pair have opposite orders. Thus every ordered type appears once in the full period. □

The word (1) has no fully separated term. Lemma 2 is a multiplicity statement, not a construction of polygons or a realization theorem in higher n. The four term axioms alone do not require a fully separated term. The reference to that term in the polygonal request must therefore be distinguished from an explicit normalized-input hypothesis. We do not prove the additional polygonal construction requirements in either interpretation.

4 Literature and verification boundaries

The stronger definition was already explicit in Dhandapani–Goodman–Holmsen–Pollack [3, Section 2, p. 5]: each directed separating and supporting type occurs once per full period. Novick's published definition [4, pp. 1060–1061] requires two supporting and two separating switches per body pair per half-period. Both definitions exclude (1). The missing multiplicity condition is established literature, not a new definition.

The Oberwolfach contribution immediately attributes an affirmative answer to Habert and Pocchiola [2, pp. 2492–2493]. Their final published Theorem 49 [5, p. 624] realizes an already specified valid fibration of an arrangement of pseudocircles. Its following application identifies the properly constrained interval encodings with fibrations. Theorem 1 concerns the weaker four term axioms; it supplies no counterexample to that representation theorem. We do not independently reprove the representation theorem or the full coding equivalence for the stronger input class.

The contribution here is the explicit finite obstruction to the literal four-axiom universal assertion and the boundary lemma above. The inspected definitions are those of the dated author manuscript and the official 2008 report. The final Goodman–Pollack journal body and an older 2004 version cited in the early representation literature were not obtained; no assertion about a correction or its absence in those versions is made. The documented priority search is bounded. It certifies neither historical firstness of this observation nor that a corresponding question remains open today.

The accompanying source and verification archive preserves the frozen candidate artifacts and independent reviews. Separate exact enumerations check all $6^4 = 1296$ possible first halves for two bodies. They agree on 48 rooted primitive weak-axiom words: 16 satisfy switch-once and 32 have no mixed event. These are indexed words, not isomorphism-class counts. The archive

also checks invalid inputs, the symmetry transforms of (1), positive separated-start controls and artifact integrity. The counterexample and Lemma 2 do not depend on the enumeration. Finite software checks do not certify a topological representation proof, literature completeness or formal proof-assistant verification.

References

[1] J. E. Goodman and R. Pollack, *The combinatorial encoding of disjoint convex sets in the plane*, author manuscript, 26 November 2006, 13 pp. `https://math.nyu.edu/inmemoriam/pollack/pubs/dblperm11-26-06.pdf`. References here use this manuscript's numbering.

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[3] R. Dhandapani, J. E. Goodman, A. Holmsen and R. Pollack, *Interval sequences and the combinatorial encoding of planar families of pairwise disjoint convex sets*, author manuscript, 21 April 2005, 19 pp., Section 2. `https://math.nyu.edu/inmemoriam/pollack/pubs/interval.pdf`. The journal citation is Rev. Roumaine Math. Pures Appl. **50** (2005), 537–553; page references here use the manuscript.

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[5] L. Habert and M. Pocchiola, *LR Characterization of Chirotopes of Finite Planar Families of Pairwise Disjoint Convex bodies*, Discrete Comput. Geom. **50** (2013), 552–648. doi:10.1007/s00454-013-9532-y. The fibration definition and Theorem 49 are on pp. 623–624.