

All-load stability of a deterministic-first-chunk file-sharing chain

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Abstract

We prove positive recurrence at every fixed positive arrival rate for the four-population deterministic-first-chunk chain described by Norros, Reittu and Eirola. This answers the stability conjecture in Norros’s 2010 Oberwolfach contribution and the retained 2011 author proof, for the exact six-rate model with a permanent seed and invisible empty waiting rooms. A single coercive Foster function combines remaining workload, rounded cohort imbalance, and two bounded finite birth–death Poisson correctors. Monotone ratio cutoffs make every first-download interface error nonpositive. The other interface errors are controlled pointwise. The proof requires no stationary averaging, moment assumption, or uniformity in the arrival rate. Exact finite controls supplement the analytic proof. The contribution is the all-load extension of the stated historical conjecture; classical methods and source-version limits are explicit.

1 The model and the question

An arriving peer chooses, independently and uniformly, which of two chunks to download first. Let a, b count empty peers assigned to the two choices, and let x, y count peers possessing just the respective chunks. A completed peer departs immediately. There is one permanent seed. The empty waiting rooms are invisible to the sampling denominator. Precisely, for $\lambda > 0$ and $D = x + y + 1$, the chain on $\mathbb{N}_0^4$ has transitions

increment in (a, b, x, y)	rate	(1)
$(1, 0, 0, 0)$	$\lambda/2$	
$(0, 1, 0, 0)$	$\lambda/2$	
$(-1, 0, 1, 0)$	$a(x+1)/D$	
$(0, -1, 0, 1)$	$b(y+1)/D$	
$(0, 0, -1, 0)$	$x(y+1)/D$	
$(0, 0, 0, -1)$	$y(x+1)/D$.	

A decrement at a zero coordinate has rate zero. These are the rates of Figure 2 in [2] and the checked 2011 author proof of [3]; the seed produces the +1 terms.

Norros [1, pp. 2782–2783] conjectured stochastic stability despite instability of the corresponding large-system differential equations. The retained January 5, 2011 author proof states the stability conjecture in Section 3.2 as Conjecture 3.6. The earlier 2009 preprint [2, Conjecture 3.4] conjectured high-load instability instead. These editions must be distinguished. Stability here means irreducibility and positive recurrence in continuous time, as in those sources.

Theorem 1. *For every fixed real $\lambda > 0$, the chain (1) is nonexplosive, irreducible and positive recurrent. It therefore has a unique stationary probability distribution.*

The exact rates are part of the theorem. In particular, no bound on the seed's aggregate upload rate is imposed beyond (1). We assert no uniform bound in λ , finite stationary population moment, geometric ergodicity, or conclusion for other chunk-selection protocols. The classical Foster method is background [4]. The work below is the all-load pointwise certificate for this chain.

Write $\mathbf{n} = a + b + x + y$. A short observation already handles small loads. For the workload $W = 2a + 2b + x + y$, direct calculation gives

$$\mathcal{L}W = 2\lambda - \frac{a(x+1) + b(y+1) + 2xy + x + y}{D} \leq 2\lambda - \frac{\mathbf{n}}{\mathbf{n} + 1}. \quad (2)$$

Thus the classical drift criterion gives stability when $0 < \lambda < 1/2$. The existence of some stable parameter values, even with an unstable large-system limit, does not require our all-load argument. Our aim is the extension to every positive load.

2 Workload, imbalance, and a finite corrector

For the rest of the proof fix $\lambda > 0$ and put

$$Z = a + x - b - y, \quad u = \frac{a(x+1) + b(y+1)}{D}, \quad d = \frac{2xy + x + y}{D}.$$

First downloads leave Z unchanged. Its other jumps have size one and total rate $\lambda + d$. Hence

$$\mathcal{L}W = 2\lambda - u - d, \quad \mathcal{L}Z = \frac{y - x}{D}. \quad (3)$$

For $m > 0$ define the rounded absolute value

$$h_m(z) = \begin{cases} z^2/(2m) + m/2, & |z| \leq m, \\ |z|, & |z| \geq m. \end{cases}$$

Its derivative is z/m clipped to $[-1, 1]$, and is globally $1/m$ -Lipschitz. Therefore, also across the rounding boundaries,

$$h_m(z + v) - h_m(z) \leq h'_m(z)v + \frac{v^2}{2m}.$$

Set

$$\beta = \frac{17\lambda}{8}, \quad q = \frac{7}{8}, \quad c = 2(\beta + 4), \quad m = 4c, \quad \theta = \beta + c + 4.$$

Equations (3) give

$$\mathcal{L}(W + ch_m(Z)) \leq \beta - u - qd + ch'_m(Z)\frac{y - x}{D}. \quad (4)$$

The last term has absolute value at most c . If $Z \geq m$ and $x \geq 3(y + 1)$, it is at most $-c/2$.

Put

$$\gamma = \frac{\theta + 1}{\theta + 2}, \quad \eta = \frac{1 + \gamma}{2}, \quad \ell = \frac{1}{\eta - \gamma}.$$

Consider on $\{0, \dots, R\}$ the reflected birth-death generator Q_R with births $\gamma(k + 1)$ for $k < R$ and deaths k for $k > 0$. Its stationary weights are γ^k and its mean is

$$\mu_R = \frac{\sum_{k=0}^R k\gamma^k}{\sum_{k=0}^R \gamma^k}.$$

The means increase to $\gamma/(1-\gamma) = \theta + 1$. We can therefore choose a finite integer $R \geq 1$ such that

$$\mu := \mu_R > \theta, \quad q(R+1) > \beta + c + 2. \quad (5)$$

For $0 \leq k < R$ put

$$t_k = \frac{\sum_{j=0}^k \gamma^j (j - \mu)}{\gamma^{k+1} (k+1)}, \quad g(k) = - \sum_{j=k}^{R-1} t_j,$$

and let $g(k) = 0$ for all integers $k \geq R$.

Lemma 1. *The function g is nonnegative and decreasing. With $G = g(0) > 0$,*

$$Q_R g(k) = k - \mu \quad (0 \leq k \leq R).$$

For the actual chain,

$$\mathcal{L}g(y) \leq y - \mu \quad \text{if } 0 \leq y \leq R, \quad b/D \geq \gamma. \quad (6)$$

For $y > R$ all increments of $g(y)$ vanish. The same assertions hold with (b, y) replaced by (a, x) .

Proof. The mean restricted to the prefix $\{0, \dots, k\}$ is strictly below μ , so $t_k < 0$. This proves monotonicity and nonnegativity. Since $g(k+1) - g(k) = t_k$, subtracting successive prefix sums gives the Poisson equation for $k < R$. The full mean-zero sum gives the terminal equation

$$Rg(R-1) = R - \mu.$$

In the actual chain the birth rate of y is $(b/D)(y+1)$ and its death rate is $y(x+1)/D \leq y$. Increasing a birth rate multiplies a nonpositive increment of g ; decreasing a death rate multiplies a nonnegative increment. This proves (6), including $y = R$, whose upward increment is zero. At $y = R+1$ the downward successor is R , where g is also zero; this proves the last assertion. $\square$

3 A single cutoff function and its interfaces

Let f be zero on $[0, \gamma]$, linear with slope ℓ on $[\gamma, \eta]$, and one on $[\eta, \infty)$. Define

$$J_X = f(b/D)g(y), \quad J_Y = f(a/D)g(x), \quad J = J_X + J_Y, \quad V = W + ch_m(Z) + J. \quad (7)$$

Thus $0 \leq J \leq 2G$ and $V \geq W \geq n$.

Lemma 2. *At every state,*

$$\mathcal{L}J \leq f(b/D)\mathcal{L}g(y) + f(a/D)\mathcal{L}g(x) + \frac{\ell G(\lambda + 4d)}{D}, \quad (8)$$

$$\mathcal{L}J \leq G(\lambda + 2d). \quad (9)$$

Proof. For a transition $s \rightarrow s'$ the exact product identity is

$$\Delta J_X = f(b/D) [g(y') - g(y)] + g(y') [f(b'/D') - f(b/D)]. \quad (10)$$

There is an identical expression for J_Y . A first download increases D and does not increase either waiting-room numerator. Both cutoff increments are nonpositive, so their interface contributions in (10) are nonpositive, regardless of a, b . An arrival changes only one cutoff, by at most ℓ/D .

A departure has $D \geq 2$ and decreases D by one. For either numerator w ,

$$0 \leq f\left(\frac{w}{D-1}\right) - f\left(\frac{w}{D}\right) \leq \frac{2\ell}{D}. \quad (11)$$

Indeed the difference is zero if $w/D \geq \eta$; otherwise $w < \eta D$ and the Lipschitz bound is $\ell w/[D(D-1)] \leq \ell\eta/(D-1) \leq 2\ell/D$. There are two cutoff terms, and the post-jump g values are at most G . Summing their interface bounds over the rates gives (8). There are no departures at $D = 1$.

For (9), a first download makes each whole product in (7) nonincreasing: its cutoff decreases and its affected g value also decreases. An arrival increases J by at most G , and a departure increases it by at most $2G$. Sum over their total rates λ, d . $\square$

4 Negative drift in all states outside a finite set

Choose a finite integer N satisfying

$$\begin{aligned} N &\geq R + 2, & N &\geq 3(R + 1), \\ (1 - \eta)N - (1 + \eta)R - \eta &\geq m, \\ N &\geq \ell G[\lambda + 4(2R + 1)]. \end{aligned}$$

All constants are finite and $1 - \eta > 0$.

If $\min(x, y) \geq R + 1$, both corrections and every one-step successor correction vanish. Also $d \geq \min(x, y)$: for $x \geq y$, $D(d - y) = y(x - y) + x \geq 0$. Equations (4)–(5) imply

$$\mathcal{L}V \leq \beta + c - q(R + 1) < -2. \quad (12)$$

Next suppose $y \leq R$ and $x \geq N$. The correction involving $g(x)$ and all its successors vanish. We have

$$d \leq 2R + 1, \quad \frac{x - y}{D} \geq \frac{1}{2}, \quad qd \geq y. \quad (13)$$

For the last inequality, $x \geq 3(y + 1)$ gives $x/D \geq 3/4$, so $qd \geq (21/32)(2y + 1) \geq y$. The remainder in (8) is at most one by the choice of N . Lemma 1, including the case $f = 0$, gives

$$f(b/D)\mathcal{L}g(y) \leq f(b/D)(y - \mu).$$

If $b/D \geq \eta$, the cutoff is one and

$$\mathcal{L}V \leq \beta - u - qd + c + y - \mu + 1 \leq \beta + c - \mu + 1 < -3. \quad (14)$$

If $b/D < \eta$, then

$$Z = a + x - b - y \geq (1 - \eta)x - (1 + \eta)y - \eta \geq m.$$

The last term in (4) is therefore at most $-c/2$. Since $f(b/D)(y - \mu) \leq y$ and $qd \geq y$,

$$\mathcal{L}V \leq \beta - u - c/2 + 1 = -3 - u \leq -3. \quad (15)$$

This includes the entire cutoff ramp. Interchanging the two types handles the opposite corner.

In every remaining state $x + y \leq N + R$. Since $d \leq x + y$, (4) and (9) give

$$\mathcal{L}V \leq \beta + c + G[\lambda + 2(N + R)] - u, \quad u \geq \frac{a + b}{N + R + 1}.$$

With

$$B = (N + R + 1)\{\beta + c + G[\lambda + 2(N + R)] + 1\},$$

we conclude

$$\mathcal{L}V \leq -1 \quad \text{outside } F = \{x + y \leq N + R, a + b \leq B\}. \quad (16)$$

The set F is finite. This is one globally defined certificate, rather than a choice between unaccounted local drift functions.

5 The continuous-time recurrence conclusion

We give the recurrence step to make its clock and integrability requirements explicit. Arrivals form a rate- λ Poisson process. Every internal event decreases W by one and every arrival increases it by two. Thus the total number of jumps up to time t is at most $W(0) + 3P_\lambda(t)$, proving nonexplosion. Every state has a positive-rate finite path to zero, because the seed makes all required downloads and departures possible. Conversely, arrivals followed by prescribed first downloads build any finite state, proving irreducibility.

Since $|Z| \leq W$ and $h_m(z) \leq |z| + m/2$, we have

$$W \leq V \leq (1 + c)W + cm/2 + 2G. \quad (17)$$

Every jump of V is bounded in absolute value by a finite constant depending on λ . Localize Dynkin's identity at the first hit τ_F of F , time t , and the first exit from a finite population set. Equation (16) and $V \geq 0$ bound the expected stopped time by $V(s)$. Nonexplosion and Fatou remove the population cutoff; monotone convergence as $t \rightarrow \infty$ yields

$$\mathbb{E}_s \tau_F \leq V(s), \quad s \notin F. \quad (18)$$

No invariant law or stationary moment was used.

For completeness, select from every state of finite F a finite positive-rate path to zero. These finitely many paths have a common time horizon T and a positive lower success probability. Starting in F , the bound $W(T) \leq W(0) + 2P_\lambda(T)$ and (17) give a uniform finite bound on $\mathbb{E}V(S_T)$. Equation (18) therefore bounds uniformly the expected duration of a trial of length T followed by return to F . The strong Markov property and the uniform positive success probability give finite expected hitting time of zero by repeated trials. The mean holding time at zero is $1/\lambda$; each of its two exit neighbors has finite expected hitting time back to zero. The resulting regeneration cycle has finite mean. Its normalized occupation measure is stationary, and irreducibility gives uniqueness. This proves Theorem 1.

6 Credit, verification, and disclosure

The model, its conjecture, the distinction between stochastic and large-system stability, and the standard Foster principle are prior work [1, 2, 3, 4]. The finite geometric birth-death Poisson equation and Lipschitz rounding estimate are standard tools. The certificate above combines them with cutoffs whose first-download interface terms have a useful sign.

Oguz, Anantharam and Norros [5] prove stability of their Common Chunk protocol using samples of three from the entire population and a rule protecting departure of rare chunks. That protocol is different from (1). For example, with just the seed and one peer possessing chunk one, its departure probability per unit contact time is $7/8$, whereas (1) has departure rate $1/2$.

A fixed arrival rate also prevents identifying the chains by a state-dependent clock change. Their final remaining rare-chunk conjecture concerns a different algorithm and is not evidence that the deterministic-first-chunk conjecture remained open.

An extensive bounded priority review identified no earlier exact all-load recurrence theorem in the sources inspected. The complete 2012 Allerton body was compared with this generator, including its operative proof. The final 2011 publisher body, the final 2015 journal extension of the 2012 paper, and some later bodies remain uninspected. Their absence from the checked evidence is not proof that they contain no relevant result. The supplement records actual reading and access limits. No worldwide-firstness or continuing-openness certificate is asserted. The source target is the stated 2010 conjecture and the checked 2011 author-proof model.

The supporting materials preserve the original proof and exact rational control programs, fresh independent boundary and counterexample controls, expected outputs, source-version notes and file hashes. They check finite states and identities; the all-state, all-real-load theorem is proved analytically above. These controls are not a proof-assistant certificate. No stationary moment or performance bound is inferred from them.

AI tools were used extensively in the research, proof development, literature comparison, computational reproduction, manuscript preparation and adversarial auditing. This is an unrefereed preprint draft and has not undergone formal independent human peer review. Internal AI reviews do not constitute such human peer review.

References

- [1] I. Norros, On the stability of population processes (joint work with H. Reittu), in *Mathematical Challenges in Stochastic Networks*, Oberwolfach Reports 48/2010, pp. 2782–2783. doi:10.4171/OWR/2010/48.
- [2] I. Norros, H. Reittu and T. Eirola, *On the stability of two-chunk file-sharing systems*, arXiv:0910.5577v1 (2009). <https://arxiv.org/abs/0910.5577v1>.
- [3] I. Norros, H. Reittu and T. Eirola, On the stability of two-chunk file-sharing systems, *Queueing Systems* **67** (2011), 183–206. doi:10.1007/s11134-011-9209-2. Model and conjecture checked in the January 5, 2011 author proof, <https://hal.science/hal-00781341>.
- [4] S. P. Meyn and R. L. Tweedie, Stability of Markovian processes III: Foster–Lyapunov criteria for continuous-time processes, *Advances in Applied Probability* **25** (1993), 518–548. doi:10.2307/1427522.
- [5] B. Oguz, V. Anantharam and I. Norros, Stable, distributed P2P protocols based on random peer sampling, in *50th Annual Allerton Conference on Communication, Control, and Computing* (2012), pp. 915–919.