A type-D_4 counterexample to the canonical-potential exponent converse

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17 October 2026 Version 1.0

Abstract

The exponent-bound converse stated for canonical GHKK potentials in Schumann's 2021 Oberwolfach report is false. For an explicit reduced word in type D_4, the canonical potential has 22 combined monomials, all of coefficient one, and all 22 inequalities define distinct facets, although two monomials have exponent -2 in one canonical coordinate. The reduced word and its complete string-coordinate support occur in earlier public computations; we extract the consequence for the historical formulation and supply an independent canonical calculation and exact integer certificates. The chamber-ansatz pullback has all exponents in $\{-1, 0, 1\}$, so the later string-coordinate converse is not refuted. No first-priority or current-openness claim is made.

Status. Extensively AI-assisted, unrefereed research note. There has been no external human peer review or proof-assistant certification.

1 The historical assertion and the result

Schumann's report [1, printed p. 1125] formulates a converse for the canonical potential W restricted to the dual cluster torus designated by a reduced word $\mathbf{i}$ for the longest Weyl group element. In that formulation, the monomial inequalities are irredundant precisely when every coordinate exponent of W has absolute value at most one. The correspondence with the string cone is unimodular [3, 2]. An invertible monomial change preserves irredundancy, but need not preserve that exponent bound.

We use the reduced-word seed and $\mathcal{X}$-mutation conventions of [2, Eq. (3), Defs. 7–9], including Dynkin adjacency in the inclined-arrow rule as specified in [3, §4.2.2]. These are the designated canonical coordinates, not an arbitrary change chosen to create a large exponent.

Theorem 1. *Let $G = \mathrm{Spin}_8(\mathbb{C})$, with Dynkin edges $1-2$, $2-3$, $2-4$, and*

$$\mathbf{i} = (1, 2, 3, 2, 4, 2, 1, 2, 3, 2, 4, 2).$$

In its canonical reduced-word $\mathcal{X}$-coordinates, W has 22 distinct monomials, each with coefficient one. Their 22 tropical inequalities define distinct facets of a full-dimensional cone in $\mathbb{R}^{12}$. Two monomials have X_6-exponent -2. Consequently the necessary direction of the canonical-W converse in [1] fails.

The word and its complete chamber-pullback support already occur in the earlier computational data [4, Case 267]; see Section 5. We do not claim first computation of these objects. The theorem is confined to the historical canonical-coordinate assertion. It does not refute the later ς-coordinate converse or coefficient conjecture in [2].

2 The canonical potential

Put $y_k = X_k^{-1}$. The frozen positions are $7, 9, 11, 12$. The complete arrow set, with all multiplicities one, is

$$\begin{gathered} 1 \to 7, \quad 2 \to 4, \quad 3 \to 2, \quad 3 \to 9, \quad 4 \to 6, \quad 5 \to 4, \quad 5 \to 11, \quad 6 \to 1, \\ 6 \to 8, \quad 7 \to 6, \quad 8 \to 3, \quad 8 \to 10, \quad 9 \to 8, \quad 10 \to 5, \quad 10 \to 12, \quad 11 \to 10. \end{gathered}$$

For completeness, let k^+ be the next occurrence of i_k, or 13. Horizontal arrows are $k \to k^+$ when $k^+ \le 12$. There is an inclined arrow $\ell \to k$ for $k < \ell < k^+ < \ell^+$ when i_k, i_ℓ are Dynkin neighbors. Arrows between frozen vertices are omitted.

The inversion roots, in the ordered simple-root basis, are

$$\begin{gathered} (1,0,0,0), (1,1,0,0), (1,1,1,0), (0,0,1,0), \\ (1,1,1,1), (1,1,0,1), (1,2,1,1), (0,1,1,0), \\ (0,1,1,1), (0,0,0,1), (0,1,0,1), (0,1,0,0). \end{gathered}$$

These are all twelve positive D_4 roots, and the product of the simple reflections is $-I_4$, so the word is reduced for w_0.

Set $b_{ab} = \#(a \to b) - \#(b \to a)$. Mutation at a mutable position k gives

$$y_k' = y_k^{-1}, \qquad y_a' = \begin{cases} y_a (1+y_k)^{b_{ak}} & b_{ak} \ge 0, \ a \ne k, \\ y_a y_k^{-b_{ak}} (1+y_k)^{b_{ak}} & b_{ak} < 0, \ a \ne k. \end{cases} \tag{1}$$

The exchange matrix changes by sign reversal in row/column k and otherwise

$$b_{ab}' = b_{ab} + [b_{ak}]_+ [b_{kb}]_+ - [-b_{ak}]_+ [-b_{kb}]_+,$$

with frozen–frozen entries erased. All right-hand sides use the old matrix. An optimized frozen position has only incoming mutable arrows; its canonical summand is the inverse frozen coordinate in that seed [2, Def. 7].

Lemma 1. *Writing*

$$A = 1 + y_3 + y_2 y_3, \qquad B = y_2 y_3 y_4 y_5,$$

the canonical potential is

$$\begin{aligned} F_7 &= 1 + y_6 + y_6 y_8 A + y_6 y_8 y_{10} \big((1+y_5) A + B (1 + y_6 + y_1 y_6) \big), \\ F_9 &= 1 + y_8 + y_8 y_{10} (1 + y_5 + y_4 y_5), \\ W &= y_7 F_7 + y_9 F_9 + y_{11} (1 + y_{10}) + y_{12}. \end{aligned} \tag{2}$$

Proof. The following fixed-label sequences, read left to right, optimize the indicated frozen position:

f	mutation sequence
7	$(2, 6, 1, 3, 4, 8, 5, 10)$
9	$(4, 8, 5, 10)$
11	(10)
12	$()$

Every entry is mutable. In each case the final frozen row has its only nonzero entry $b_{f,10} = -1$, as follows by the displayed matrix rule. Thus the final seed is optimized, and the canonical summand is y_f there.

Pull back that monomial, in reverse order, by (1). For $f = 7$, after the reverse steps $10, 5, 8, 4$ the quotient by y_7 is

$$1 + y_8 + y_8y_{10} + y_5y_8y_{10} + y_4y_5y_8y_{10}.$$

The next step, at 3, gives

$$1 + y_8 + y_3y_8 + y_8y_{10}(1 + y_5 + y_3 + y_3y_5 + y_3y_4y_5).$$

The step at 1 adds $y_1y_3y_4y_5y_8y_{10}$. After the step at 6 it is

$$1 + y_6 + y_6y_8(1 + y_3) + y_6y_8y_{10}(1 + y_5 + y_3 + y_3y_5 + y_3y_4y_5) + (1 + y_1)y_3y_4y_5y_6^2y_8y_{10}.$$

The final step at 2 gives F_7. The four-step $f = 9$ pullback gives F_9; the last two summands are immediate. These exact rational identities identify the canonical normalized potential, rather than merely a polynomial defining the same cone. □

The summands in (2) contain $14, 5, 2, 1$ distinct terms. Every term has exactly one frozen marker, and the coefficients are all one. In particular

$$y_2y_3y_4y_5y_6^2y_7y_8y_{10}, \qquad y_1y_2y_3y_4y_5y_6^2y_7y_8y_{10}$$

occur. They have X_6-exponent -2.

3 An exact certificate for every facet

Write the 22 terms as $y^{a^{(j)}}$. In X-cocharacter coordinates t the inequality is $-a^{(j)} \cdot t \geq 0$. Put $z = -t$; the cone becomes

$$C = \{z \in \mathbb{R}^{12} : a^{(j)} \cdot z \geq 0, \quad 1 \leq j \leq 22\}.$$

The following table gives its complete exponent support and integer vectors $z^{(j)}$. Each exponent row is encoded as twelve consecutive single digits in position order $1, \ldots, 12$; no entry has more than one digit.

j	$a^{(j)}$	$z^{(j)}$
1	000000000001	$(0,0,0,0,0,0,0,0,0,0,0,-1)$
2	000000000010	$(0,0,0,0,-1,0,0,0,0,1,-1,0)$
3	000000000110	$(0,0,0,0,0,0,0,1,0,-1,0,0)$
4	000000001000	$(0,0,-1,0,0,0,0,1,-1,0,0,0)$
5	000000011000	$(-2,0,0,0,0,1,0,-1,0,1,0,0)$
6	000000011100	$(-2,0,0,0,1,1,0,0,0,-1,1,0)$
7	000000100000	$(-1,0,0,0,0,1,-1,0,0,0,0,0)$
8	000001100000	$(0,2,0,-1,0,-1,0,1,0,0,0,0)$
9	000001110000	$(0,0,1,-1,0,0,0,-1,1,1,0,0)$
10	000001110100	$(0,0,1,-1,1,0,0,0,1,-1,1,0)$
11	000010011100	$(-2,0,0,1,-1,1,0,0,0,0,0,0)$
12	000011110100	$(0,0,1,0,-1,0,0,0,1,0,0,0)$

j	$a^{(j)}$	$z^{(j)}$
13	000110011100	$(0,1,0,-1,0,0,0,0,0,0,0,0)$
14	001001110000	$(0,1,-1,-1,0,0,0,0,0,1,0,0)$
15	001001110100	$(-1,1,-1,-1,1,1,0,0,1,-1,1,0)$
16	001011110100	$(-1,1,-1,0,-1,1,0,0,1,0,0,0)$
17	011001110000	$(0,-1,0,0,0,0,0,0,1,0,0)$
18	011001110100	$(-1,-1,0,0,1,1,0,0,1,-1,1,0)$
19	011011110100	$(-1,-1,0,1,-1,1,0,0,1,0,0,0)$
20	011111110100	$(0,0,0,-1,-1,1,0,0,2,0,0,0)$
21	011112110100	$(1,0,0,0,0,-1,1,0,0,0,0,0)$
22	111112110100	$(-1,0,0,0,0,0,0,0,0,0,0,0)$

Direct integer multiplication of the displayed rows and columns gives

$$a^{(j)} \cdot z^{(j)} = -1, \qquad a^{(r)} \cdot z^{(j)} \geq 0 \quad (r \neq j). \tag{3}$$

This is a complete finite certificate: no optimization tolerance or unprovided historical computation is a premise. Omitting the jth inequality admits $z^{(j)}$, proving irredundancy.

There is also a strict facet certificate. Set

$$u = (0,0,0,0,0,0,1,0,1,0,1,1).$$

Every exponent row has $a^{(j)} \cdot u = 1$, so C has a strict interior point and is full-dimensional. For $p_j = z^{(j)} + u$,

$$a^{(j)} \cdot p_j = 0, \qquad a^{(r)} \cdot p_j \geq 1 \quad (r \neq j).$$

A neighborhood of p_j in the hyperplane $(a^{(j)})^{\perp}$ satisfies all the other inequalities strictly. Thus the selected face has dimension 11. The certificates separate every pair of selected inequalities, so the 22 facets are distinct. Together with Lemma 1, this proves Theorem 1.

4 The chamber map and the scope boundary

Let H be the D_4 Cartan matrix. The chamber-ansatz map of [2, Def. 9] can be written

$$y_k = \prod_{\ell=1}^{12} x_\ell^{K_{k\ell}}, \qquad K_{k\ell} = \begin{cases} 1 & \ell = k \text{ or } \ell = k^+ \leq 12, \\ H_{i_k,i_\ell} & k < \ell < k^+, \\ 0 & \text{otherwise.} \end{cases}$$

The upper-triangular matrix K has diagonal one, hence determinant one. A monomial row a pulls back to aK. For every one of the 22 rows in the table, all entries of aK lie in $\{-1,0,1\}$. In particular rows 21, 22 give, respectively,

$$(0,1,0,1,0,1,-1,0,0,0,0,0), \qquad (1,0,0,0,0,0,0,0,0,0,0,0).$$

The monomial isomorphism therefore preserves the inequality presentation but does not preserve its coordinatewise exponent bound.

The later ς-coordinate converse in [2, Conj. 3] is not refuted: this example satisfies its exponent condition. Nor is its separate coefficient criterion refuted, since all coefficients here are one and all inequalities are essential. The computation contradicts the assertion in the proof of [2, Thm. 2, p. 1046] that the two exponent bounds are equivalent under the chamber map. It does not contradict that theorem's conclusion, and supplies no general replacement proof for its affected step. Our direct facet proof does not use that assertion or that theorem.

5 Prior computations and reproducibility

The primary-author dataset [4, Case 267] already records this word and its complete $14+1+5+2$ string-coordinate support. The relevant Git commit is dated 6 March 2020; a Git date is provenance metadata, not independent proof of the date of public availability. The repository is cited by the later computational work [5], providing a public pre-PR citation trail. We attribute the earlier word and support computations accordingly. The published minuscule nonredundancy results [1, 2] also cover the three leaf-root summands; the central summand here is a single monomial.

The contribution of this note is to spell out the consequence for the specified historical canonical-W assertion, independently reconstruct the canonical potential, and give complete directly checkable facet certificates. A bounded literature audit found no earlier explicit statement of this refutation, but cannot establish first priority, current openness, or exhaustive coverage of inaccessible or unindexed sources. No claim is made that the original converse remained open in October 2026.

The accompanying verification archive includes the manuscript source, the complete exponent and certificate data, a Python standard-library verifier that reconstructs the seed and optimized pullbacks over exact integer Laurent arithmetic, and a separate SymPy reconstruction using two reduced-word braid-graph searches. That reviewer inherited candidate context; its freeze before a local file comparison does not establish blindness. A further reviewer received only the word, Dynkin diagram and primary references in a fresh context, froze its support before comparison, and checked all 12,504 directed braid-graph edges and 22 separate facet certificates. Its replay and original input-boundary receipt are included. Source examples, mutation involutivity and deliberately corrupted inputs provide implementation controls. The written formulas and displayed certificates, rather than sampled numerical experiments, establish the result. Downloaded source documents, third-party programs, and the original unlicensed dataset are not redistributed.

AI use and review status. AI tools were used extensively to develop, solve, reconstruct, verify and adversarially review the arguments, search the literature, and prepare this note and its code. Independent internal AI reviews are not external human peer review. This preprint is unrefereed and is not a formal proof-assistant certificate.

References

[1] B. Schumann (joint work with G. Koshevoy), Optimality of string cone inequalities and potential functions, *Oberwolfach Report* 20/2021, pp. 1123–1125. doi:10.4171/OWR/2021/20.

[2] G. Koshevoy and B. Schumann, Redundancy in string cone inequalities and multiplicities in potential functions on cluster varieties, *J. Algebraic Combin.* **56** (2022), 1031–1053. doi:10.1007/s10801-022-01144-z.

[3] V. Genz, G. Koshevoy and B. Schumann, Polyhedral parametrizations of canonical bases & cluster duality, *Adv. Math.* **369** (2020), 107178. doi:10.1016/j.aim.2020.107178.

[4] D. Mironov, *crystal*, public computational repository, `crystal_D_4.out.xz`, Case 267. Pinned commit `0d6e564cf5e69d79c1074d5d1e3e72ac700904d6` (commit dated 6 March 2020). https://github.com/mironovd/crystal.

[5] G. Koshevoy and D. Mironov, F-polynomials and Newton polytopes, *EasyChair Preprint* 8975 (3 October 2022). https://easychair.org/publications/preprint/z55N.