

Failure of strong quadratic growth in unregularized sparse control of a critical wave equation

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Abstract

We give an explicit counterexample to retaining ordinary critical-cone strict positivity and strong control- L^2 quadratic growth after removing quadratic control regularization from the energy-critical quintic-wave problem of Kunisch and Meinschmidt. On the three-dimensional unit ball, zero is the unique global optimizer for a sufficiently small positive pointwise control cap. Its ordinary critical cone is $\{0\}$, so the strict condition is vacuous, but feasible controls converging in L^∞ have objective gap $o(\|u\|_{L^2}^2)$. Nearby radial terminal targets admit global optimizers, and every such optimizer has L^2 displacement divided by target displacement tending to infinity. Local L^1 quadratic growth survives. The construction realizes the established concentration and noncoercivity mechanism in the actual nonlinear source model, using classical wave estimates.

1 Question, conventions and precise result

Kunisch–Meinschmidt’s wave-control analysis [1, Theorem 4.17] proves strong control- L^2 quadratic growth from strict positivity on the ordinary critical cone when the quadratic control coefficient β_2 is positive. The Oberwolfach contribution [2], printed pp. 438–440, places the $\beta_2 = 0$ question immediately after this implication. We answer the question of retaining that implication unchanged. Adapted sufficient conditions, enlarged cones and weaker growth metrics remain distinct questions. The counterexample uses the zero critical cone explicitly.

Let B be the unit ball in $\mathbb{R}^3$, $H = L^2(B)$ and $T = 1$. All controls and states are real. The state is the unique energy/Strichartz mild solution of

$$y_{tt} - \Delta y + y^5 = u, \quad y|_{\partial B} = 0, \quad (y(0), y_t(0)) = (0, 0). \quad (1)$$

Put $H^{-1} = H^{-1}(B)$, with the dual norm of $H_0^1(B)$, and define

$$\begin{aligned} J_{d,\nu}(u) &= \frac{1}{2} \|y_u(1) - d\|_H^2 + \frac{\nu}{2} \|(y_u)_t(1)\|_{H^{-1}}^2 \\ &\quad + \frac{1}{4} \|y_u\|_{L^4(0,1;L^{12}(B))}^4 + \|u\|_{L^1(0,1;H)}, \quad 0 \leq \nu < \infty, \\ \mathcal{U}_\rho &= \{u \in L^2(0,1;H) : \|u(t)\|_H \leq \rho \text{ a.e.}\}. \end{aligned} \quad (2)$$

Thus $\gamma = \beta_1 = 1$ and $\beta_2 = 0$. The velocity term is the optional term in [2]; $\nu = 0$ gives the objective in [1]. Throughout, $\|u\|_1$ and $\|u\|_2$ denote time-space $L^1(H)$ and $L^2(H)$ norms; $\|u\|_\infty$ denotes $L^\infty(0,1;H)$. The normalized radial eigenfunction is

$$e(x) = \frac{\sin(\pi|x|)}{\sqrt{2\pi}|x|}, \quad e(0) = \frac{\pi}{\sqrt{2\pi}}, \quad \|e\|_H = 1, \quad -\Delta e = \pi^2 e. \quad (3)$$

Theorem 1. *There is $\rho_* > 0$ such that, for every $0 < \rho \leq \rho_*$ and every fixed finite $\nu \geq 0$, the problem (2) with $d = \pi e$ has the following properties.*

1. *Zero is its unique global optimizer. With $A = \|u\|_1$,*

$$J_{\pi e, \nu}(u) - J_{\pi e, \nu}(0) \geq \frac{A^3}{12\rho^2} \quad (u \in \mathcal{U}_\rho). \quad (4)$$

2. *The ordinary critical cone at zero is $C(0) = \{0\}$. The source's substitute second-variation strict condition on $C(0) \setminus \{0\}$ holds vacuously.*

3. *The feasible controls $u_h = h e \mathbf{1}_{|t-1/2| < h}$, for $0 < h < \min(\rho, 1/2)$, satisfy*

$$\|u_h\|_\infty = h, \quad \|u_h\|_2^2 = 2h^3, \quad J_{\pi e, \nu}(u_h) - J_{\pi e, \nu}(0) = \left(\frac{\pi^2}{3} + \frac{2}{\pi^2}\right) h^4 + O(h^6). \quad (5)$$

Consequently no positive control- L^2 quadratic-growth constant exists in either an L^2 or an L^∞ neighborhood of zero.

4. *For $d_\varepsilon = (\pi + \varepsilon)e$, global optimizers exist and every choice of global optimizers satisfies*

$$\frac{\|u_\varepsilon\|_2}{\varepsilon} \longrightarrow \infty \quad (\varepsilon \downarrow 0). \quad (6)$$

The cap ρ_* and the constants in (4) can be chosen independently of ν . Small neighborhoods in the radial perturbation assertion may depend on the fixed ν .

The distinction between exact-cone positivity and coercivity against normalized weakly null sequences is established in second-order control theory [4, 6, 7, 8]. The Dunn example, as reproduced in [5, Example 4.16], already illustrates exact-cone insufficiency. Group-sparse threshold calculus is classical [3]. Our contribution is the explicit realization with a genuine nonlinear-wave optimizer and the quantitative assertions above. An extensive bounded primary-source audit located no earlier covering realization; it does not certify worldwide firstness. The wave estimates used below are standard inputs from [1].

2 Uniform small-input estimates and global optimality

Use the compatible state space

$$\mathcal{Y} = \{y \in C([0, 1]; H_0^1(B)) : y_t \in C([0, 1]; H), y \in L^4(0, 1; L^{12}(B))\},$$

with the sum norm and the distributional derivative compatibility. Let $Lu = z_u$ denote the zero-data linear-wave solution. The bounded-domain energy/Strichartz estimate in [1, Lemma 2.9] gives $\|Lu\|_{\mathcal{Y}} \leq K\|u\|_1$. Sobolev embedding and interpolation give

$$\|y\|_{L^5 L^{10}}^5 \leq C\|y\|_{L^4 L^{12}}^4 \|y\|_{L^\infty H_0^1}. \quad (7)$$

Lemma 2. *There are $a_0, E, R > 0$ such that, for every u with $A = \|u\|_1 \leq a_0$,*

$$\|y_u\|_{\mathcal{Y}} \leq EA, \quad \|y_u - z_u\|_{\mathcal{Y}} \leq RA^5. \quad (8)$$

The constants are independent of the terminal target and ν .

Proof. On the closed compatible zero-data subspace of $\mathcal{Y}$, consider $y \mapsto L(u - y^5)$. By (7), the quintic has norm at most $M\|y\|_{\mathcal{Y}}^5$ in $L^1(H)$ and difference norm at most $M(\|y\|_{\mathcal{Y}} + \|w\|_{\mathcal{Y}})^4\|y - w\|_{\mathcal{Y}}$. For $A \leq a_0$ sufficiently small this map preserves the radius- $2KA$ ball and is a contraction there. The resulting mild solution agrees with the unique energy/Strichartz state in [1]. Its norm is at most $2KA$, and $y_u - z_u = -L(y_u^5)$ proves the second bound. Enlarge E, R as necessary to bound terminal position in H and terminal velocity in H^{-1} . The compatible residual $y + L(y^5) - Lu$ has derivative the identity in y at zero. Its quintic part is a bounded polynomial by (7), so the local implicit-function theorem also gives a smooth state map with $S'(0) = L$ and $S''(0) = 0$. This direct argument does not require an inverse onto arbitrary continuous position/velocity pairs. $\square$

For $u \in \mathcal{U}_\rho$, write $f(t) = \|u(t)\|_H$, $A = \int f$ and $r = \|u\|_2$. Then

$$0 \leq A \leq \rho, \quad A \leq r, \quad r^2 \leq \rho A. \quad (9)$$

Projection of the linear state onto (3) gives

$$\langle e, z_u(1) \rangle = \frac{1}{\pi} \int_0^1 \sin(\pi t) \langle e, u(t) \rangle dt. \quad (10)$$

This is only a projection identity; the nonlinear state need not stay in the span of e . Expanding the terminal square, discarding nonnegative terms, and using (8) yield

$$\Delta(u) := J_{\pi e, \nu}(u) - J_{\pi e, \nu}(0) \geq \int_0^1 (1 - \sin \pi t) f(t) dt - C_0 A^5, \quad (11)$$

where C_0 is fixed independently of ν .

Since $1 - \sin(\pi t) \geq 2(t - 1/2)^2$, a same-mass comparison gives

$$\int_0^1 (t - 1/2)^2 f(t) dt \geq \frac{A^3}{12\rho^2}. \quad (12)$$

Indeed, put $b = A/(2\rho) \leq 1/2$ and $f_* = \rho \mathbf{1}_{|t-1/2| < b}$. The product $((t - 1/2)^2 - b^2)(f - f_*)$ is nonnegative on both the centered interval and its complement. Its integral proves (12), because $\int(f - f_*) = 0$. The cases $A = 0$ and $A = \rho$ are included. Choose $\rho_* \leq a_0$ with $C_0 \rho_*^4 \leq 1/12$. Equations (11)–(12) then give

$$\Delta(u) \geq \frac{A^3}{6\rho^2} - C_0 A^5 \geq \frac{A^3}{12\rho^2}.$$

This proves global uniqueness in Theorem 1. No numerical value for an unknown PDE constant is asserted.

3 Ordinary cone and loss of strong growth

In the source convention, the ordinary critical cone consists of tangent directions v with zero first directional derivative of the whole cost. At zero, bounded truncations of any $v \in L^1(H)$ or $L^2(H)$ are radially feasible after sufficiently small scaling. They converge in the respective norm, so the closed tangent cone is the whole space. Pointwise constraint slack alone would not imply norm interiority in these two spaces. The adjoint and sparse subgradient are

$$p(t) = -\sin(\pi t)e, \quad \lambda(t) = \sin(\pi t)e, \quad \|\lambda(t)\|_H \leq 1,$$

with zero constraint multiplier. Direct differentiation gives

$$DJ_{\pi e, \nu}(0; v) = \int_0^1 (\|v(t)\|_H - \sin(\pi t) \langle e, v(t) \rangle) dt \geq \int_0^1 (1 - \sin \pi t) \|v(t)\|_H dt. \quad (13)$$

The right-hand side is positive for every nonzero Lebesgue equivalence class: its weight vanishes only at $t = 1/2$. Hence $C(0) = \{0\}$. The source explicitly sets the substitute sparse $j''(0; v^2)$ to zero; this is not a classical second derivative of a norm. Its strict condition on nonzero critical directions is satisfied vacuously.

For the pulse in Theorem 1, direct linear-wave integration gives

$$A_h = 2h^2, \quad r_h^2 = 2h^3, \quad z_{u_h}(1) = \frac{2h \sin(\pi h)}{\pi^2} e, \quad (z_{u_h})_t(1) = 0. \quad (14)$$

Its linear tracking-plus-sparse gap is

$$2h^2 - \frac{2h \sin(\pi h)}{\pi} + \frac{2h^2 \sin^2(\pi h)}{\pi^4} = \left(\frac{\pi^2}{3} + \frac{2}{\pi^2} \right) h^4 + O(h^6).$$

Lemma 2 bounds the nonlinear terminal error by $O(h^{10})$, the quartic state cost by $O(h^8)$, and the velocity cost by $O(\nu h^{20})$. This proves (5). Dividing by $2h^3$ proves the claimed failure of growth, even with $\|u_h\|_\infty \rightarrow 0$.

For comparison with the established NDC framework [7, 8], set $v_h = u_h/r_h$. These unit-norm $L^2(H)$ directions are weakly null: their pairing with an L^2 test is bounded by the test's norm on the shrinking interval. With F the smooth costs and $G = \|\cdot\|_1 + I_{\mathcal{U}_\rho}$, the normalized second-order expression is

$$\begin{aligned} \frac{G(u_h) - G(0) + F'(0)u_h}{r_h^2} + \frac{1}{2}F''(0)[v_h, v_h] &= \frac{1}{h} - \frac{\sin(\pi h)}{\pi h^2} + \frac{\sin^2(\pi h)}{\pi^4 h} \\ &= \left(\frac{\pi^2}{6} + \frac{1}{\pi^2} \right) h + O(h^3) \rightarrow 0. \end{aligned}$$

Thus exact-cone positivity misses this familiar weak-null obstruction. The direct pulse argument already handles both source localizations.

4 Existence and terminal-target perturbations

We record existence for the optional velocity term as well. The set $\mathcal{U}_\rho$ is closed, convex and bounded in $L^2(H)$, hence weakly compact. Along a minimizing sequence, Lemma 2 bounds the states uniformly in energy and $L^4 L^{12}$. Compactness of $H_0^1 \hookrightarrow H$ and the bound on y_t imply strong compactness of positions in $C([0, 1]; H)$. Moreover $y_{tt} = \Delta y - y^5 + u$ is bounded in $L^\infty(H^{-1})$: $y \in L^\infty L^6$, so $y^5 \in L^\infty L^{6/5} \hookrightarrow L^\infty H^{-1}$. Compactness of $H \hookrightarrow H^{-1}$ then gives strong compactness of velocities in $C([0, 1]; H^{-1})$.

A subsequence of states converges almost everywhere. The bounded $L^{6/5}$ quintics converge weakly to the quintic of the limit; the limit also retains the weak energy and $L^4 L^{12}$ bounds. Consequently its quintic lies in $L^1(H)$ by (7). Passing to the distributional equation and initial traces identifies a zero-data weak solution. Subtract the linear Duhamel solution forced by $u - y^5$. The remaining homogeneous Dirichlet wave has zero initial data; projection onto a complete Dirichlet eigenbasis proves it is zero. Thus the limit is the actual mild state, by uniqueness. Endpoint costs are continuous under the two strong convergences, and the sparse and quartic norms are weakly

lower semicontinuous. The direct method therefore gives global optimizers for every $d \in H$ and every finite $\nu \geq 0$ in this capped small-input setting.

Fix $a = \rho/2$ and use the feasible trial

$$\hat{u}_\varepsilon = ae \mathbf{1}_{|t-1/2| < \pi\varepsilon/(2a)}.$$

For sufficiently small ε , its mass is $\pi\varepsilon$, its linear endpoint is $\varepsilon e + O(\varepsilon^3)$ and its linear terminal velocity is zero. Expansion as above gives, for fixed ν ,

$$J_{d_\varepsilon, \nu}(\hat{u}_\varepsilon) - J_{d_\varepsilon, \nu}(0) = -\frac{1}{2}\varepsilon^2 + O(\varepsilon^3) \leq -\frac{1}{4}\varepsilon^2. \quad (15)$$

For all feasible u , discarding nonnegative costs gives instead

$$J_{d_\varepsilon, \nu}(u) - J_{d_\varepsilon, \nu}(0) \geq -\|g_\varepsilon\|_{L^2(0,1)}r - C_1A^5, \quad g_\varepsilon(t) = [(1 + \varepsilon/\pi)\sin(\pi t) - 1]_+. \quad (16)$$

For small ε ,

$$g_\varepsilon(t) \leq [\varepsilon/\pi - 2(t - 1/2)^2]_+, \quad \|g_\varepsilon\|_{L^2} \leq 2^{1/4}\pi^{-5/4}\varepsilon^{5/4}. \quad (17)$$

If an optimizer had $r \leq K\varepsilon$, then $A \leq r$ and (16) would be $-o(\varepsilon^2)$, contradicting (15). For every fixed K this is impossible for sufficiently small ε , proving (6) for every optimizer.

5 A surviving weaker-norm estimate

Corollary 3. *For the same example there is $\delta > 0$ with*

$$\Delta(u) \geq \frac{\|u\|_1^2}{4\pi^2} \quad (u \in \mathcal{U}_\rho, \|u\|_1 \leq \delta). \quad (18)$$

For any target $\pi e + w$, with $\varepsilon = \|w\|_H$ sufficiently small, every global optimizer satisfies

$$\|u\|_1 \leq 4\pi^2 E\varepsilon, \quad \|u\|_2 \leq 2\pi\sqrt{\rho E}\sqrt{\varepsilon}. \quad (19)$$

The constants and this upper-bound neighborhood can be chosen uniformly for all finite $\nu \geq 0$. On $w = \varepsilon e$, for each fixed ν ,

$$\|u_\varepsilon\|_1 = \Theta(\varepsilon), \quad c\varepsilon^{3/4} \leq \|u_\varepsilon\|_2 \leq C\sqrt{\varepsilon} \quad (20)$$

for every global optimizer and sufficiently small ε . The lower-bound neighborhood in (20) may depend on ν .

Proof. Set $x = \langle e, y_u(1) \rangle \leq b = A/\pi + RA^5$. The full tracking square gives $\Delta(u) \geq q_A(x) = A - \pi x + x^2/2$. Choose δ so that $b \leq \pi/2$ and $\pi R\delta^3 \leq 1/(4\pi^2)$. Since q_A decreases for $x < \pi$,

$$\Delta(u) \geq q_A(b) = \frac{A^2}{2\pi^2} - \pi RA^5 + \frac{R}{\pi}A^6 + \frac{R^2}{2}A^{10} \geq \frac{A^2}{4\pi^2}.$$

For a perturbed global optimizer, comparison with zero gives $\Delta(u) \leq \langle w, y_u(1) \rangle \leq E\varepsilon A$. First (4) implies $A \leq \sqrt{12\rho^2 E\varepsilon}$, placing it in the δ neighborhood. Then (18) yields the first upper bound, and $r^2 \leq \rho A$ gives the second. Every step discards the nonnegative velocity cost, so these upper conclusions are uniform in ν .

On the radial ray, the upper $A = O(\varepsilon)$ makes the last term in (16) $O(\varepsilon^5)$. Combining (15) with (17) gives $r \geq c\varepsilon^{3/4}$. Also $\|g_\varepsilon\|_\infty \leq \varepsilon/\pi$, so the same comparison gives $A \geq c'\varepsilon$. This proves (20), with the smallness threshold inherited from the fixed- ν trial comparison. $\square$

These are estimates relative to the fixed zero optimizer. They assert no pairwise Lipschitz selection, perturbed optimizer uniqueness, L^∞ convergence, or sharpness of the exponent bracket. Weaker-norm and strengthened-cone theories are developed in [9, 6, 7, 8]; the elementary estimate here uses the terminal tracking square and the cap. For example, taking target zero in the same model gives $J_{0,\nu}(u) \geq \|u\|_1 \geq \|u\|_2^2/\rho$. Thus the theorem is a specific obstruction to the unchanged implication, not universal failure of all unregularized quadratic growth.

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