

Coarse endpoint concentration at every diverging scale for one-dimensional space-time random walks

Alec Kriebel

Independent researcher

ORCID: 0009-0001-9320-500X

18 October 2026

Abstract

For a nearest-neighbor walk on $\mathbb{Z}$ in a fixed i.i.d. space-time environment with transition probabilities strictly between zero and one, we prove that its quenched and averaged endpoint laws agree, after coarse graining into deterministic intervals of any diverging length, outside an environment event of probability smaller than every inverse power of time. No uniform ellipticity or quantitative rate of divergence of the interval length is required. This is the affirmative one-dimensional assertion contrary to Conjecture 1 in Berger’s contribution to Oberwolfach Report 38/2009. The proof combines replica overlap estimates, environmental martingales, same-parity coalescence and concentration over a short final segment. **Priority qualification:** novelty and historical priority remain unresolved. A bounded literature audit established no exact full anticipation, but four relevant older original full texts could not be accessed. We make no claim of firstness, independent discovery or that the conjecture remained open until this work. Established late-layer and coalescing methods, including those of Dunlap–Gu and Barraquand–Rychnovsky, are explicitly credited.

1 Statement, interpretation and prior work

Let $\omega = (p_{s,x} : s \geq 0, x \in \mathbb{Z})$ have i.i.d. coordinates with

$$0 < p_{s,x} < 1 \quad \mathbb{P}\text{-almost surely.} \quad (1)$$

In a fixed environment the walk starts at $X_0 = 0$ and at time s jumps from x to $x + 1$ with probability $p_{s,x}$ and to $x - 1$ otherwise. Write P^ω for this quenched law, $\mu_n^\omega(x) = P^\omega(X_n = x)$, and $\alpha_n(x) = \mathbb{E}\mu_n^\omega(x)$. The environment expectation and probability are $\mathbb{E}$ and $\mathbb{P}$. We suppress ω in intermediate calculations. For a deterministic partition $\mathcal{P}$ of $\mathbb{Z}$ define

$$L_{n,\mathcal{P}} = \sum_{I \in \mathcal{P}} |\mu_n(I) - \alpha_n(I)|.$$

This uses the full ℓ^1 distance, twice the usual total variation distance. For real $M > 0$ and $b \in \mathbb{R}$, the intervals $[b + jM, b + (j + 1)M)$, $j \in \mathbb{Z}$, intersected with $\mathbb{Z}$, define a partition; empty cells are harmless.

Theorem 1. *For every fixed environment law satisfying (1), every deterministic sequence $M_N > 0$ with $M_N \rightarrow \infty$, and every $\varepsilon, K > 0$, there are finite $C_{\varepsilon,K}$ and N_0 such that, for $N \geq N_0$ and every deterministic alignment b_N ,*

$$\mathbb{P}(L_{N,\mathcal{P}_N} > \varepsilon) \leq C_{\varepsilon,K} N^{-K}, \quad \mathcal{P}_N = \{[b_N + jM_N, b_N + (j + 1)M_N) \cap \mathbb{Z} : j \in \mathbb{Z}\}. \quad (2)$$

The constants may depend on the fixed environment law. The eventual threshold may also depend on the sequence M_N . The bound is uniform in each deterministic alignment.

Here and below *superpolynomially small* means $O(N^{-K})$ for every fixed $K > 0$, with its own constant and eventual threshold. The theorem includes biased averaged walks, nearest-neighbor parity, integer cells containing M_N sites, real interval lengths and arbitrarily slowly diverging nonmonotone scales. It does not assert a simultaneous bound for environment-selected partitions, a law-uniform bound as ellipticity degenerates, or a result for static or correlated environments.

Berger’s report [1, pp. 2151–2152, Theorem 1 and Conjecture 1] states the analogous concentration assertion in transverse dimension at least two and conjectures its failure in dimension one. Theorem 1 gives the affirmative assertion for its i.i.d., nearest-neighbor, elliptic one-dimensional model. A deterministic elliptic law has $\mu_N = \alpha_N$; the theorem also covers every nondegenerate law under (1), so its conclusion is not based on that trivial special case.

The difference-walk and quenched-mean mechanisms already occur in Balázs–Rassoul-Agha–Seppäläinen [2]. Independent particles and coupling in dynamical environments are treated in Joseph–Rassoul-Agha–Seppäläinen [5]. Dunlap–Gu [3] isolate a short final layer in a Gaussian-times-stationary-corrector local limit theorem for a continuum stochastic flow with molecular Brownian noise. Barraquand–Rychkovsky [4, Section 1.4, Proposition 1.13 and Section 4.5] prove a full-space beta-environment local limit theorem and use final-layer decomposition and ordered coalescence. Those mechanisms, as well as replica calculations, martingale exposure and bounded differences, are established methods. All analytic inputs needed here are proved below.

The modern distribution-valued fluctuation limits of Drillick–Parekh [6] and Parekh [7] are also relevant. The inspected statements concern different observables and convergence topologies and do not state (2)’s shrinking-cell ℓ^1 conclusion with a superpolynomial environment tail. Nearest-neighbor parity is not a reason to exclude that framework: Parekh [7, Section 6.1, Remark 6.2] handles it by two-step kernels on $2\mathbb{Z}$. A qualitative local or weak limit does not, by itself, supply every inverse-power probability bound.

The bounded audit verified no earlier theorem with the full quantifiers of (2); it did not establish novelty. The complete original texts of Boldrighini–Minlos–Pellegrinotti (1999) [8], Boldrighini–Pellegrinotti (2001) [9], Boldrighini–Minlos–Pellegrinotti (2004) [10], and Bérard (2004) [11] could not be accessed. Their metadata, abstracts or later primary discussions were available, but do not exclude stronger estimates in the original proofs. None of these four sources is represented as fully eliminated as an anticipation. The accompanying priority ledger records exact access and reading limits. Accordingly this note reports a verified self-contained theorem with originality and historical priority unresolved; it does not assert firstness, independent discovery, a new method or present-day open status.

Put

$$q = \mathbb{E}p, \quad a = q(1 - q), \quad \delta = \mathbb{E}[p(1 - p)], \quad \sigma^2 = \text{Var}(p) = a - \delta, \quad v = 2q - 1. \quad (3)$$

Then $0 < \delta \leq a \leq 1/4$. The averaged walk has independent sign increments of mean v and variance $4a$. This follows by successively averaging fresh time layers even when a path revisits an old spatial location. Constants $C, c > 0$ below may change from line to line and depend on the fixed law. Let $\mathcal{F}_s$ contain the environment layers with time index less than s , and put $I_s = \sum_x \mu_s(x)^2$.

2 Replica returns and occupation moments

Lemma 2. *There is $C < \infty$ such that, for all deterministic integers $s, t \geq 0$,*

$$\mathbb{E}[I_{s+t} \mid \mathcal{F}_s] \leq \frac{C}{\sqrt{t+1}}. \quad (4)$$

For $L, k \geq 1$ integers,

$$\mathbb{E} \left[\left(\sum_{j=0}^{L-1} I_{s+j} \right)^k \mid \mathcal{F}_s \right] \leq k! C^k L^{k/2}. \quad (5)$$

Consequently, for every fixed $\eta > 0$, $L \leq N$ and deterministic s , the event $\sum_{j < L} I_{s+j} > N^\eta \sqrt{L}$ is superpolynomially unlikely.

Proof. Take two independent walks conditional on the same environment. Their half-difference $D = (X - Y)/2$ is integral because the positions have the same parity. On environmental averaging its transition probabilities to each neighbor are a away from zero and δ at zero; its holding probabilities are $1 - 2a$ and $1 - 2\delta$, respectively. Indeed different current sites use independent fresh variables, whereas at a shared site the probabilities of the two opposite step pairs are both $\mathbb{E}[p(1-p)]$. In particular,

$$\mathbb{E}I_t = P_0(D_t = 0) =: g_t.$$

We derive a pointwise return bound. For $0 < z < 1$, the hitting transform from $d \geq 1$ to zero is $\rho(z)^d$, where the root in $(0, 1)$ solves

$$az\rho^2 - [1 - (1 - 2a)z]\rho + az = 0.$$

The other root exceeds one; the bounded solution of the hitting recurrence is the stated one. First-return decomposition at zero gives

$$\begin{aligned} G(z) &:= \sum_{t \geq 0} g_t z^t = \frac{1}{1 - (1 - 2\delta)z - 2\delta z \rho(z)} \\ &= \frac{1}{(1 - \delta/a)(1 - z) + (\delta/a)\sqrt{(1 - z)(1 - (1 - 4a)z)}} \\ &\leq \frac{\sqrt{a}}{2\delta\sqrt{1 - z}}. \end{aligned} \quad (6)$$

The square root is positive on $(0, 1)$; the last inequality uses $\delta \leq a$ and $1 - (1 - 4a)z \geq 4a$.

To pass from this generating-function bound to individual coefficients, use reversibility with weights $\pi(0) = a/\delta$ and $\pi(d) = 1$ for $d \neq 0$. The transition operator P is a self-adjoint contraction on $\ell^2(\pi)$: contraction follows from Jensen's inequality and invariance of π . Every holding probability is at least $1/2$, so $2P - \text{Id}$ is also a self-adjoint Markov contraction. Thus the spectrum of P lies in $[0, 1]$. The spectral measure of $\mathbf{1}_{\{0\}}$, divided by $\pi(0)$, is a probability measure ν with $g_t = \int_{[0,1]} \lambda^t d\nu(\lambda)$. Therefore g_t decreases with t . For $n \geq 1$ this implies

$$g_n \frac{1 - z^{n+1}}{1 - z} \leq G(z).$$

Set $z = n/(n+1)$, note $z^{n+1} \leq e^{-1}$, and apply (6). After including $n = 0$, we obtain $g_n \leq C/\sqrt{n+1}$.

We also need the backward return profile $u_t(d) = P_d(D_t = 0)$ to be maximal at zero. It is symmetric. Write $b_t(d) = u_t(d) - u_t(d+1)$ for $d \geq 0$. The backward transition equation gives

$$\begin{aligned} b_{t+1}(0) &= (1 - 2\delta - a)b_t(0) + ab_t(1), \\ b_{t+1}(d) &= ab_t(d-1) + (1 - 2a)b_t(d) + ab_t(d+1), \end{aligned} \quad d \geq 1.$$

All coefficients are nonnegative, since $1 - 2\delta - a \geq 1 - 3a \geq 1/4$. Initially $b_0(0) = 1$ and all other $b_0(d) = 0$. Induction proves $u_t(d) \leq u_t(0) = g_t$. Conditioning on $\mathcal{F}_s$ now fixes the starting replica law $\mu_s \otimes \mu_s$ and leaves later layers fresh, so

$$\mathbb{E}[I_{s+t} \mid \mathcal{F}_s] = \sum_{x,y} \mu_s(x) \mu_s(y) u_t((x-y)/2) \leq g_t.$$

Every half-difference here is integral.

Let $A_L = \sum_{j < L} g_j \leq C\sqrt{L}$. Expand the k th power in (5) and order its time indices, overcounting by at most $k!$. In each resulting nonnegative ordered sum, condition at its penultimate selected time. The sum of conditional expectations of the last overlap is at most A_L . Iterating removes all indices and costs at most A_L^k . Repeated indices are included: at a zero time gap use $I_t \leq 1 = g_0$. This proves (5) without independence of overlaps. Markov's inequality gives $k!C^k N^{-k\eta}$ for the stated tail. For each desired power choose a fixed sufficiently large k . $\square$

3 Environmental martingales and coarse expectation

Let $T_t f(x)$ be the averaged expectation of f for a t -step walk started at x . For deterministic bounded f , the exposure martingale for $\mu_n(f)$ is $\mu_s(T_{n-s}f)$, $0 \leq s \leq n$. Its next increment is exactly

$$D_s(f) = \sum_x \mu_s(x)(p_{s,x} - q)[T_{n-s-1}f(x+1) - T_{n-s-1}f(x-1)]. \quad (7)$$

If $h_s(x)$ denotes the bracket, independence within the next layer gives

$$|D_s(f)| \leq \sup_x |h_s(x)|, \quad \mathbb{E}[D_s(f)^2 \mid \mathcal{F}_s] = \sigma^2 \sum_x \mu_s(x)^2 h_s(x)^2. \quad (8)$$

All sums at a fixed horizon have finite support. For a $1/r$ -Lipschitz f , the translation semigroup preserves its Lipschitz constant; thus the increment bound is $2/r$ and the predictable quadratic variation is at most $Cr^{-2} \sum_{s < n} I_s$.

We use two elementary scalar concentration estimates, recorded here to fix their conditional meaning. If martingale differences satisfy $|D_j| \leq b$ and $Q = \sum_j \mathbb{E}[D_j^2 \mid \mathcal{G}_{j-1}]$, then

$$\mathbb{P} \left(\left| \sum_j D_j \right| \geq u, \quad Q \leq w \right) \leq 2 \exp \left\{ -\frac{u^2}{4(w + bu)} \right\}. \quad (9)$$

Indeed $e^x \leq 1 + x + x^2$ for $|x| \leq 1$ makes $\exp(\lambda \sum_{i \leq j} D_i - \lambda^2 Q_j)$ a nonnegative supermartingale for $\lambda b \leq 1$. Markov's inequality on the joint event, with $\lambda = u/[2(w + bu)]$, and the same argument for negative sums prove (9). We never condition the martingale law on a future small-variation event.

We will also use the following bounded-increment estimate. For independent centered increments with range length at most two,

$$\mathbb{E} e^{\lambda Z} \leq e^{\lambda^2/2}.$$

To check this, the second derivative of the log moment-generating function is a variance under an exponential tilt, bounded by one for any variable supported on an interval of length two. Integrate twice, using the zero mean. The corresponding exponential supermartingale, stopped at the first crossing and then bounded at the terminal time, gives

$$\mathbb{P}\left(\max_{t \leq h} |S_t| \geq u\right) \leq 2e^{-u^2/(2h)} \quad (10)$$

for centered partial sums of such independent increments. The same argument, applied to conditional ranges of martingale increments, gives the independent-input bounded-differences inequality

$$\mathbb{P}(F - \mathbb{E}F > u) \leq \exp\left\{-\frac{2u^2}{\sum_j c_j^2}\right\} \quad (11)$$

when altering input j changes F by at most c_j . For completeness, exposing independent inputs yields martingale increments of conditional range at most c_j ; the preceding log-moment argument after rescaling gives $\mathbb{E}(e^{\lambda D_j} \mid \mathcal{G}_{j-1}) \leq e^{\lambda^2 c_j^2/8}$. Multiplication and optimization prove (11).

Lemma 3. *Uniformly in real interval lengths $M \geq 1$, deterministic alignments, $n \geq 2$ and $B \geq 1$,*

$$\mathbb{E}L_{n,\mathcal{P}} \leq C \sqrt{\left(\frac{B}{M} + n^{-1/2}\right) [1 + \log(1 + \min(M+1, \sqrt{n}))]} + \frac{C}{B^2}. \quad (12)$$

In particular there is a deterministic bound $\rho(n, M) \leq 2$, uniform in alignment, with $\rho(n, M) \rightarrow 0$ whenever both $n, M \rightarrow \infty$.

Proof. On its parity lattice the averaged kernel is the binomial mass $b_t(j) = \mathbb{P}(\text{Bin}(t, q) = j)$. For $\phi(\theta) = 1 - q + qe^{i\theta}$,

$$|\phi(\theta)|^2 = 1 - 4a \sin^2(\theta/2), \quad |\phi(\theta)|^t \leq e^{-2at\theta^2/\pi^2}, \quad |\theta| \leq \pi.$$

Fourier inversion, with and without the factor $|e^{i\theta} - 1| \leq |\theta|$, proves

$$\max_j b_t(j) \leq C(t+1)^{-1/2}, \quad \max_j |b_t(j+1) - b_t(j)| \leq C(t+1)^{-1}.$$

Unimodality and zeros outside the support give $\sum_j |b_t(j+1) - b_t(j)| = 2 \max_j b_t(j)$. These are physical shifts by two, not one.

For $h_I(x) = T_t \mathbf{1}_I(x+1) - T_t \mathbf{1}_I(x-1)$, summing over the contiguous parity sites of an interval telescopes to at most two endpoint masses. Alternatively bound its at most $M+1$ site terms by the adjacent-mass bound. The partition cannot increase the ℓ^1 norm of a signed measure, so

$$\begin{aligned} \max_I |h_I(x)| &\leq C \min((t+1)^{-1/2}, (M+1)(t+1)^{-1}), \\ \sum_I |h_I(x)| &\leq C(t+1)^{-1/2}, \\ \sum_I h_I(x)^2 &\leq C \min((t+1)^{-1}, (M+1)(t+1)^{-3/2}). \end{aligned}$$

Orthogonality of (7) in time and $\mathbb{E}I_s = g_s$ therefore imply

$$\sum_I \text{Var}(\mu_n(I)) \leq C \sum_{s \leq n} (s+1)^{-1/2} \min((n-s)^{-1}, (M+1)(n-s)^{-3/2}).$$

For $s \leq n/2$ the sum is at most $Cn^{-1/2}$. For $s > n/2$, bound $(s+1)^{-1/2} \leq Cn^{-1/2}$ and split the remaining sum over $k = n - s$ at $k = (M+1)^2$. This yields

$$\sum_I \text{Var}(\mu_n(I)) \leq \frac{C}{\sqrt{n}} [1 + \log(1 + \min(M+1, \sqrt{n}))]. \quad (13)$$

The window $[vn - B\sqrt{n}, vn + B\sqrt{n}]$ intersects at most $2B\sqrt{n}/M + 2$ cells. Cauchy–Schwarz and (13) bound the expected discrepancy on them by the square-root term in (12). All other cells lie outside this window. Their expected discrepancy is at most twice its averaged tail, which is at most C/B^2 by the variance $4an$. For fixed B , the square-root term tends to zero as $n, M \rightarrow \infty$: if $M \leq \sqrt{n}$, use $\log(1+M)/M \rightarrow 0$; otherwise use $\log n/\sqrt{n} \rightarrow 0$. Then let $B \rightarrow \infty$. The infimum of the displayed bounds over $B \geq 1$, capped at two, defines a suitable ρ . $\square$

4 A polynomial auxiliary coarse scale

The next estimate is used only at an auxiliary scale, not at the possibly much slower target scale M_N .

Proposition 4. *For fixed $\gamma > 1/4$, $u > 0$ and $K > 0$,*

$$\mathbb{P}(L_{n,\mathcal{P}} > u) = O(n^{-K})$$

uniformly over integer cell lengths $M \geq n^\gamma$ and all deterministic alignments.

Proof. It suffices to treat $1/4 < \gamma < 1/2$, since a fixed exponent in that range also covers all larger γ . Put

$$d = \gamma - \frac{1}{4}, \quad \beta = \frac{1}{4} + \frac{d}{2}, \quad \eta = \frac{d}{4}, \quad r = n^\beta, \quad h = \left\lfloor \frac{n^{2\beta}}{(\log n)^2} \right\rfloor, \quad m = n - h.$$

Eventually $1 \leq h \leq n/2$. Average just the final h layers: $\nu_n = \mathbb{E}[\mu_n \mid \mathcal{F}_m] = \mu_m T_h$. For any deterministic $|f| \leq 1$, the gradient bound in the preceding proof gives

$$|D_s(f)| \leq C/\sqrt{h}, \quad Q_f \leq C \sum_{s < m} \frac{I_s}{n-s}.$$

Divide these indices into deterministic consecutive dyadic blocks B_j on which $2^j h \leq n-s-1 < 2^{j+1} h$. Their lengths are at most $h_j = 2^j h$, and there are $O(\log n)$ nonempty blocks. By Lemma 2, the common event

$$E_n = \left\{ \sum_{s \in B_j} I_s \leq n^\eta \sqrt{h_j} \text{ for all } j \right\}$$

has superpolynomially small complement. A shorter last block can be bounded by extending its deterministic interval to length h_j . On E_n , $Q_f \leq Cn^\eta/\sqrt{h}$. Equation (9), applied to the joint event, yields, for each fixed $t > 0$,

$$\mathbb{P}(|\nu_n(f) - \alpha_n(f)| > t, E_n) \leq 2 \exp\{-c_t n^{\beta-\eta}/\log n\}. \quad (14)$$

The window $W_n = [vn - \sqrt{n} \log n, vn + \sqrt{n} \log n]$ meets $J \leq Cn^{1/2-\gamma} \log n + 2$ cells. On those complete cells the ℓ^1 discrepancy is the maximum over their 2^J deterministic sign tests, each bounded by one. Union bound (14) over the signs and charge $\mathbb{P}(E_n^c)$ once. Since

$$\beta - \eta - (1/2 - \gamma) = 5d/4 > 0,$$

the negative exponential dominates $J \log 2$. The averaged mass outside W_n is at most $Ce^{-c(\log n)^2}$ by (10) (or its terminal bound). The expectation of $\nu_n(W_n^c)$ is this same mass. Markov's inequality therefore controls the remaining cells with a superpolynomial fixed-threshold tail. We have proved

$$\mathbb{P}(\|\nu_n(\mathcal{P}) - \alpha_n(\mathcal{P})\|_1 > t) \text{ is superpolynomially small.} \quad (15)$$

We restore the final layers by controlling boundary crossings. Represent the integer cells as real intervals with boundary grid $b + M\mathbb{Z}$. Let

$$B_n = \{x \in \mathbb{Z} : \text{dist}(x + vh, b + M\mathbb{Z}) \leq r\}.$$

Choose the deterministic periodic majorant

$$f_B(x) = \max\{0, \min\{1, 2 - \text{dist}(x + vh, b + M\mathbb{Z})/r\}\}.$$

It is $1/r$ -Lipschitz, equals one on B_n , and vanishes outside the radius- $2r$ boundary strips. Those strips are eventually disjoint because $r/M \rightarrow 0$ uniformly for $M \geq n^\gamma$.

For a unimodal probability sequence (b_j) and integer $D \geq 1$, its mass on one residue class modulo D is at most $D^{-1} + 2 \max b_j$. To see this, discard at most two sampled points next to the mode and compare each remaining sampled value with the average of a disjoint D -term block toward the mode, where the values are larger. The sum of these block averages is at most D^{-1} . A physical residue class modulo M for $-m + 2j$ corresponds, when nonempty, to one binomial-index class modulo $M/\gcd(2, M)$. Its mass is therefore at most $2/M + 2 \max b_j$. The support of f_B has at most $C(r + 1)$ physical residues in one period, uniformly in the real shift. The peak estimate gives

$$\mathbb{E}\mu_m(f_B) \leq C(r + 1)/M + C(r + 1)/\sqrt{m} = o(1).$$

For this Lipschitz test, (8) and Lemma 2 bound the predictable variation, off a superpolynomially unlikely event, by $Cn^{\eta+1/2-2\beta}$, with increment bound $2n^{-\beta}$. For a fixed positive deviation, (9) has exponent at least

$$\frac{c}{n^{\eta+1/2-2\beta} + n^{-\beta}},$$

which grows as a positive power since $2\beta - 1/2 - \eta = 3d/4 > 0$. Thus $\mu_m(B_n)$ has a superpolynomial fixed-threshold tail.

Let $D_n^\omega = P^\omega(|X_n - X_m - vh| > r)$. By averaging fresh final layers and (10), $\mathbb{E}D_n^\omega \leq Ce^{-cr^2/h} \leq Ce^{-c(\log n)^2}$. Its fixed-threshold tail is superpolynomial by Markov's inequality. Compare the actual endpoint and an averaged final endpoint with the real position $X_m + vh$ under the same old law. If the start is outside B_n and the displacement error is at most r , the two compared positions belong to the same cell. The coupling inequality gives

$$\|\mu_n(\mathcal{P}) - \nu_n(\mathcal{P})\|_1 \leq 4\mu_m(B_n) + 2D_n^\omega + Ce^{-c(\log n)^2}.$$

Half-open endpoint conventions cause no exception because equality with a boundary is included in B_n . Combining this bound with (15) proves the proposition. $\square$

5 Final-segment smoothing and independent strips

We now prove the all-scale assertion. Set, for large N ,

$$\ell = \lfloor N^{3/4} \rfloor, \quad m = N - \ell, \quad H = \lceil N^{5/16} \rceil, \quad r = N^{3/8}, \quad R = \lceil N^{3/8} \log N \rceil, \quad \eta = \kappa = 1/32. \quad (16)$$

Here $m \geq N/2$ eventually. All partitions and intervals introduced below are deterministic.

Lemma 5. *For any prescribed family of at most CN intervals, each containing at most $10R + 10$ integer sites,*

$$\mathbb{P} \left(\max_J \mu_m(J) > N^{-\kappa} \right) \text{ is superpolynomially small.}$$

Proof. For each interval choose a $[0, 1]$ majorant equal to one on it, tapering linearly to zero over distance r on each side. It is $1/r$ -Lipschitz. The averaged binomial peak bound gives its mean at most $C(R + r + 1)/\sqrt{m} = O(N^{-1/8} \log N) = o(N^{-1/32})$. By Lemma 2, the single event $\sum_{s < m} I_s \leq N^\eta \sqrt{N}$ has superpolynomially small complement. On it the predictable variation for every majorant is at most $CN^{-7/32}$ and the increment bound is $2N^{-3/8}$. With $u = N^{-1/32}/2$, (9) bounds each joint tail by $2e^{-cN^{5/32}}$. Union bound over the prescribed intervals, charging the common overlap failure only once. $\square$

Write $K_\ell^\omega(x, \cdot)$ for the last- ℓ -step kernel in the fresh environment after time m .

Lemma 6. *For starts x, y of the same parity,*

$$\mathbb{E}_{\text{future}} \|K_\ell^\omega(x, \cdot) - K_\ell^\omega(y, \cdot)\|_1 \leq C|x - y|/\sqrt{\ell + 1}. \quad (17)$$

Both sides may be capped by two.

Proof. In each fixed environment use independent transition coins until the two paths meet and common coins thereafter. This preserves each quenched marginal. Same-parity nearest-neighbor paths cannot cross without meeting. Before meeting, environmental averaging makes the positive half-gap a homogeneous lazy walk, killed at zero, with neighbor probabilities a, a and holding probability $1 - 2a$. If $p_\ell(j)$ is its unrestricted mass from zero, the killed kernel from $d \geq 1$ to $z \geq 1$ is $p_\ell(z - d) - p_\ell(z + d)$. This follows either by reflection, which leaves holding steps unchanged, or by checking its transition recursion, zero boundary and initial values. Its survival probability is

$$\sum_{z \geq 1} [p_\ell(z - d) - p_\ell(z + d)] = \sum_{j=1-d}^d p_\ell(j) \leq Cd/\sqrt{\ell + 1}.$$

The last bound follows by Fourier inversion: $1 - 2a + 2a \cos \theta$ is nonnegative for $a \leq 1/4$ and at most $e^{-c_a \theta^2}$ on $[-\pi, \pi]$. The coupling inequality in full ℓ^1 distance supplies factor two, and $2d = |x - y|$. The case $x = y$ is zero. $\square$

Condition on the old environment and write $\mu = \mu_m$. In a fixed H -site partition $\mathcal{P}_H$, replace the conditional distribution within each cell by the conditional α_m distribution, keeping μ 's cell mass:

$$\mu'(x) = \mu(J) \frac{\alpha_m(x)}{\alpha_m(J)}, \quad x \in J, \alpha_m(J) > 0.$$

Null α_m cells also have zero μ_m mass: every reachable parity site has positive α_m mass. Omit those cells. Couple starts of μ and μ' within each cell; their distance is less than H . Lemma 6 gives

$$\mathbb{E}_{\text{future}} \|\mu K_\ell - \mu' K_\ell\|_1 \leq CH/\sqrt{\ell}, \quad \|\mu' - \alpha_m\|_1 = L_{m, \mathcal{P}_H}.$$

The latter identity is an exact cellwise calculation. Markov kernels and partitioning contract ℓ^1 distance. Moreover, convexity and Lemma 3 give

$$\mathbb{E}_{\text{future}} \|\alpha_m K_\ell(\mathcal{P}_N) - \alpha_N(\mathcal{P}_N)\|_1 \leq \rho(\ell, M_N).$$

Indeed each start in the α_m mixture only translates the deterministic target partition alignment. Thus, for

$$F = \|\mu_m K_\ell(\mathcal{P}_N) - \alpha_N(\mathcal{P}_N)\|_1,$$

we have

$$\mathbb{E}[F \mid \mathcal{F}_m] \leq L_{m, \mathcal{P}_H} + CH/\sqrt{\ell} + \rho(\ell, M_N). \quad (18)$$

Here $H/\sqrt{\ell} = O(N^{-1/16})$ and $\rho(\ell, M_N) \rightarrow 0$ without a rate assumption on M_N . Proposition 4 applies at the fixed auxiliary exponent $5/16$: $H \geq m^{5/16}$ and $m \geq N/2$.

It remains to concentrate F over the future. From a start x at time m , kill a path after its first step $t \in \{1, \dots, \ell\}$ with

$$|X_{m+t} - x - vt| > R,$$

and let $K^\dagger(x, \cdot)$ be its surviving endpoint subprobability kernel. Averaging and (10) yield, uniformly in x ,

$$\mathbb{E}_{\text{future}}[1 - K^\dagger(x, \mathbb{Z})] \leq d_N := Ce^{-cR^2/\ell} \leq Ce^{-c(\log N)^2}. \quad (19)$$

This is a maximum over the entire segment.

Partition the fresh environmental coordinates into deterministic moving strips

$$S_k = \{(t, y) : 0 \leq t < \ell, \quad kR \leq y - vt < (k+1)R\}, \quad k \in \mathbb{Z}.$$

Only finitely many reachable coordinates are needed in each strip. These disjoint input vectors are independent and independent of $\mathcal{F}_m$, including when v is irrational. An alive path from x can use only sites satisfying $y - vt \in [x - R, x + R]$. Consequently changing strip k affects $K^\dagger(x, \cdot)$ only if

$$x \in J_k = [kR - R - 1, (k+1)R + R + 1] \cap \mathbb{Z}.$$

This statement holds in both the original and the altered environment, since the allowed tube is deterministic. No environmental variable at a killed destination is consulted.

Put $w_k = \mu_m(J_k)$ and $F^\dagger = \|\mu_m K^\dagger(\mathcal{P}_N) - \alpha_N(\mathcal{P}_N)\|_1$. Changing strip k changes $F^\dagger$ by at most $2w_k$, because each affected row is a subprobability measure and the norm satisfies the reverse triangle inequality. Each integer start belongs to at most six J_k : its admissible real interval of indices has length $3 + 2/R \leq 5$. Therefore

$$\sum_k w_k \leq 6, \quad \sum_k w_k^2 \leq 6 \max_k w_k.$$

Only $O(N)$ strips can influence starts in $[-m, m]$; choose a deterministic finite index range containing them all. Their J_k contain at most $10R + 10$ integer sites. Conditional on $\mathcal{F}_m$, (11) now gives

$$\mathbb{P}(F^\dagger - \mathbb{E}[F^\dagger \mid \mathcal{F}_m] > t \mid \mathcal{F}_m) \leq \exp\{-t^2/(12 \max_k w_k)\}. \quad (20)$$

If the denominator is zero the function is constant and the tail is zero. Lemma 5 bounds $\max_k w_k$ by $N^{-1/32}$ off an old-environment event of superpolynomially small probability. Thus each fixed positive upper deviation in (20) has cost at most $e^{-ctN^{1/32}}$. The independent objects here are the strip vectors; no independence of endpoint masses or overlapping starting cones is assumed.

Finally let $D = 1 - \mu_m K^\dagger(\mathbb{Z})$. Positivity of $K_\ell - K^\dagger$ gives $|F - F^\dagger| \leq D$. By (19), for every old environment,

$$\mathbb{E}[D \mid \mathcal{F}_m] \leq d_N, \quad \mathbb{E}[F^\dagger \mid \mathcal{F}_m] \leq \mathbb{E}[F \mid \mathcal{F}_m] + d_N, \quad \mathbb{P}(D > t \mid \mathcal{F}_m) \leq d_N/t. \quad (21)$$

Proof of Theorem 1. Fix $\varepsilon > 0$. For sufficiently large N , $CH/\sqrt{\ell} + \rho(\ell, M_N) + d_N < \varepsilon/8$. By Proposition 4, outside a superpolynomially unlikely event, $L_{m, \mathcal{P}_H} \leq \varepsilon/8$. Equations (18) and (21) then give $\mathbb{E}[F^\dagger \mid \mathcal{F}_m] \leq \varepsilon/4$. Outside another superpolynomially unlikely event, Lemma 5 supplies the strip weight bound. If also $D \leq \varepsilon/4$, then $F > \varepsilon$ implies $F^\dagger - \mathbb{E}[F^\dagger \mid \mathcal{F}_m] > \varepsilon/2$. Equations (20)–(21) therefore give, for every fixed K ,

$$\mathbb{P}(F > \varepsilon) \leq O(N^{-K}) + e^{-c_\varepsilon N^{1/32}} + 4d_N/\varepsilon.$$

Both explicit errors are superpolynomial. Since $F = L_{N, \mathcal{P}_N}$, this proves (2). All uses of the target partition were in norms, contraction and the real-interval expectation estimate; the auxiliary integer partition imposes no restriction on M_N . $\square$

6 Scope, verification and disclosure

The proof uses the fixed-law inequalities $0 < \delta \leq a \leq 1/4$, fresh space-time independence and bounded nearest-neighbor jumps. It uses neither inverse moments of p or $1 - p$ nor a common positive lower support bound. Same-parity coupling is sufficient because the old quenched and averaged laws have the same reachable parity. An arrow law with $p \in \{0, 1\}$ has $\delta = 0$ and is excluded; constants and thresholds need not remain bounded as a family of elliptic laws approaches such a law.

The asymptotic claim follows from the proofs above. The accompanying verification package includes finite exact calculations of replica transitions, overlap moments, martingale identities, parity residue bounds, coalescing kernels and strip influences; such calculations check identities and edge cases and are not a substitute for the analytic argument.

AI tools were used extensively in solving, drafting and checking this work. Separate adversarial AI reviews of the underlying argument and exact verification calculations informed this note. It is unrefereed and has not undergone conventional human peer review. The author is responsible for the statements and for any remaining errors. The bounded priority audit leaves the explicit full-text access gaps described in the introduction. The manuscript does not certify a novel discovery or establish when the historical conjecture was first resolved.

References

- [1] N. Berger, *Quenched Local Limit Theorem for Space-time Random Walk in Random Environment*, pp. 2151–2152, in *Scaling Limits in Models of Statistical Mechanics*, Oberwolfach Report 38/2009. doi:10.4171/OWR/2009/38.
- [2] M. Balázs, F. Rassoul-Agha and T. Seppäläinen, *The random average process and random walk in a space-time random environment in one dimension*, Commun. Math. Phys. **266** (2006), 499–545. Author manuscript.
- [3] A. Dunlap and Y. Gu, *A quenched local limit theorem for stochastic flows*, J. Funct. Anal. (2022). doi:10.1016/j.jfa.2021.109372.
- [4] G. Barraquand and M. Rychkovsky, *Random walk on nonnegative integers in beta distributed random environment*, Commun. Math. Phys. (2023). doi:10.1007/s00220-022-04536-1; arXiv:2201.07270v3, Sections 1.4 and 4.5.

- [5] M. Joseph, F. Rassoul-Agha and T. Seppäläinen, *Independent Particles in a Dynamical Random Environment*, in *Probability and Analysis in Interacting Physical Systems*, Springer Proceedings in Mathematics & Statistics **283** (2019), 75–121. doi:10.1007/978-3-030-15338-0_4.
- [6] H. Drillick and S. Parekh, *Random walks in space-time random media in all spatial dimensions: the full subcritical fluctuation regime* (2025). arXiv:2510.22155v1.
- [7] S. Parekh, *Hierarchy of KPZ limits arising from directed random walk models in random media*, Forum Math. Sigma (2026). doi:10.1017/fms.2026.10220.
- [8] C. Boldrighini, R. A. Minlos and A. Pellegrinotti, *Central limit theorem for a random walk in dynamical environment: integral and local*, Theory Stoch. Process. **5(21)** (1999), nos. 3–4, 16–28. Original full text unavailable in the present audit.
- [9] C. Boldrighini and A. Pellegrinotti, $T^{-1/4}$ -Noise for random walks in dynamic environment on $\mathbb{Z}$, Mosc. Math. J. **1** (2001), no. 3, 365–380. doi:10.17323/1609-4514-2001-1-3-365-380. Original full text unavailable in the present audit.
- [10] C. Boldrighini, R. A. Minlos and A. Pellegrinotti, *Random walks in quenched i.i.d. space-time random environment are always a.s. diffusive* (2004). doi:10.1007/s00440-003-0331-x. Original full text unavailable in the present audit.
- [11] J. Bérard, *The almost sure central limit theorem for one-dimensional nearest-neighbour random walks in a space-time random environment* (2004). doi:10.1239/jap/1077134669. Original full text unavailable in the present audit.