

A modular-storage proof of NP-completeness for chip-firing halting on simple digraphs

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Abstract

We give a self-contained proof that ordinary conservative chip-firing halting is NP-complete on finite loopless simple strongly connected digraphs. Hardness holds with polynomial numerical total initial chips and a specified vertex whose deletion leaves an acyclic graph of depth at most one. A separate construction forbids antiparallel arcs. The proof combines the primed-toppler mechanism of Holroyd–Levine–Winkler with the classical prime/Boolean/clause encoding of Stockmeyer–Meyer, also reconstructed by Gramlich. We prove the physical conservative realization, its legal checkpoint dynamics and the arbitrary-schedule implication in full; NP membership is credited to Farrell–Levine and reproved with an explicit certificate bound. This is an attributed proof of the historical simple-digraph question. Exact full-result and realization priority and preexisting worldwide openness remain unknown after a bounded audit; no firstness is claimed. AI tools were used extensively. The paper is unrefereed and has not received external human peer review.

1 Question, conventions and attribution

An input is a finite loopless digraph $G = (V, E)$ with $N = |V| \geq 2$, at most one arc per ordered pair, strong connectivity, and a configuration $x \in \mathbb{Z}_{\geq 0}^V$ encoded in binary. A vertex v of outdegree d_v may fire when $x_v \geq d_v$: it loses d_v chips and sends one along each outgoing arc. All vertices obey this rule; there is no dissipative sink. A configuration is stable if $x_v < d_v$ for every v . The question HALTING asks whether some legal finite firing word reaches a stable configuration. Lemma 1 below supplies the usual schedule independence, without a fairness assumption. An *oriented* graph additionally has no antiparallel pair.

Farrell–Levine [1, §3.1, Table 1] explicitly posed the simple-digraph NP-completeness question while proving membership in NP and hardness for binary directed multigraphs. Their historical question is the target here; an archived open-problem label is not evidence of current worldwide openness. We establish the following exact statement.

Theorem 1. *HALTING is NP-complete under deterministic polynomial-time many-one reductions on the input class above. Hardness persists when a specified root r has the property that $G - r$ contains no directed path of two arcs and is acyclic, and when the numerical total initial chips is polynomial in the 3-SAT input length. The same assertions hold on the oriented subclass.*

The ingredients require precise credit. The literal multiplicities in Lemma 2 specialize Holroyd–Levine–Winkler’s parallel primed-toppler construction [2, Lemma 4.1]. The Boolean residues and local clause products are the classical Stockmeyer–Meyer encoding [3, Theorem 6.1], as reconstructed

by Gramlich alone [4, Fact 3], presented here in complement form. We claim neither encoding as new. Root-last stabilization and conserved-mass comparison have antecedents in Farrell’s thesis [5, §§1.1,4.1] and Farrell–Levine [1, §2]. Least action and tree periods are prior mathematics; our proof of membership specializes [1, §3]. Even exponentially long simple-graph games with polynomial initial chips are prior [6]. The argument below supplies a complete conservative graph realization and SAT equivalence without asserting priority for that realization or its oriented version.

2 Least action and a polynomial certificate

Put $\Delta_{v,v} = d_v$ and $\Delta_{v,u} = -1$ for $u \rightarrow v$, with other entries zero. A word with firing-count vector g changes x to $x - \Delta g$.

Lemma 1 (Classical least action). *If $f \in \mathbb{Z}_{\geq 0}^V$ has stable $z = x - \Delta f$, every legal word has firing counts at most f coordinatewise. In particular a nonnegative stable z is a sound certificate of halting, even if f is not given as a legal word.*

Proof. At a first firing that would exceed f_v , the current count at v is f_v and all other counts are at most their coordinates of f . Nonnegative incoming arc counts put the chips at v at most $x_v - d_v f_v + \sum_{u \rightarrow v} f_u = z_v < d_v$, contradicting legality. Thus a legal word has length at most $\sum_v f_v$. Repeatedly firing any legal vertex must stop at a stable configuration. Conversely, an infinite legal word rules out every nonnegative stabilizing vector. Early voluntary stopping at an active state is not halting. $\square$

For completeness we give the polynomial-bit certificate argument, credited to Farrell–Levine’s method. Strong connectivity and the directed matrix-tree theorem give a strictly positive primitive integer vector $\pi \in \ker \Delta$: its entries are the numbers of spanning in-arborescences toward each vertex, divided by their common gcd. At each nonroot tree vertex there are at most $N - 1$ outgoing choices, so

$$1 \leq \pi_v \leq (N - 1)^{N-1}. \quad (1)$$

If x halts, choose its actual terminating firing vector f and final stable $z \geq 0$. Some w has $f_w < \pi_w$: otherwise $f - \pi \geq 0$ has the same stable z , and Lemma 1 applied to the actual word gives $f \leq f - \pi$, a contradiction. For every arc $u \rightarrow v$, chip balance yields

$$f_u \leq \sum_{a \rightarrow v} f_a = d_v f_v + z_v - x_v \leq d_v(f_v + 1) - 1, \quad f_u + 1 \leq d_v(f_v + 1).$$

Follow a simple directed path from u to w , of length at most $N - 1$. Since $d_v \leq N - 1$ and $f_w + 1 \leq \pi_w$, we obtain

$$f_u + 1 \leq (N - 1)^{N-1} \pi_w \leq (N - 1)^{2N-2}. \quad (2)$$

Each coordinate therefore has $O(N \log N)$ bits. For $N = 2$ the only graph is the directed two-cycle; its only nonnegative stable input, by conservation, is zero, with $f = 0$, consistent with (2). A verifier checks $f \geq 0$ and $0 \leq x - \Delta f < d$ using binary arithmetic in polynomial time. This proves membership in NP for both graph classes and arbitrary input chip magnitudes, without claiming polynomial simulation time.

The additive one above is essential. For arcs $0 \rightarrow 1, 1 \rightarrow 0, 1 \rightarrow 2, 2 \rightarrow 0$, input $(1, 0, 0)$ halts after firing 0 once. The inequality $f_0 \leq d_1 f_1$ would be false. The intermediate source inequality on PDF page 10 of [1] omits this slack; the self-contained argument here retains it. This observation is about that line, not a rejection of the prior membership theorem or a certification of every other source proof.

3 An attributed modular bank

Lemma 2. Let $q \geq 2$, $\emptyset \neq A \subseteq \mathbb{Z}/q\mathbb{Z}$, and let h be its periodic indicator on $\mathbb{Z}$. For $0 \leq a < q$ put

$$c_a = 1 + h(-a - 1) - h(-a), \quad S_A(k) = \sum_{a=0}^{q-1} c_a((a + k) \bmod q).$$

Here residues lie in $\{0, \dots, q-1\}$. There are q counters in total: $c_a \in \{0, 1, 2\}$ and $\sum_a c_a = q$. For every integer k ,

$$S_A(k) = K_A + qh(k), \quad K_A = \frac{q(q-1)}{2} - |A|. \quad (3)$$

Thus its attainable maximum is $B_A = K_A + q$, attained exactly on A .

Proof. The multiplicity claims follow from $h \in \{0, 1\}$ and telescoping. Advancing k adds one to each of the q counters, except that a wrapping counter loses q . The wrapping offset is $a = -k - 1 \pmod{q}$, whence

$$S_A(k+1) - S_A(k) = q - qc_{-k-1} = q(h(k+1) - h(k)).$$

So $S_A - qh$ is constant. Over one period each counter has average $(q-1)/2$; the average of qh is $|A|$, proving (3). Nonemptiness ensures that the displayed maximum is attained. Full A gives constant storage. Even if $K_A < 0$ in a modulus-two boundary case, all actual residue counts and storage remain nonnegative. Empty A would not attain $K_A + q$ and is excluded. $\square$

This formula is an exact earlier-mechanism specialization, not a new toppler lemma. In [2, Lemma 4.1], take $F(k) = k + h(0) - h(k)$ for $k \geq 0$. Then $F(0) = 0$, $F(k+q) = F(k) + q$, and its increments are $0, 1, 2$. The lemma uses $F(i) - F(i-1) = 1 + h(i-1) - h(i)$ topplers primed at $a = q - i$; these are precisely c_a . Their cumulative topplings equal $F(k)$. A physical degree- q counter emits q chips per toppling, so conservation gives $S_A(k) = S_A(0) + qk - qF(k)$, in agreement with (3). A one-output abstract toppler or splitter does not by itself establish an ordinary conservative chip-firing implementation; the conservative-gate distinction is explicit in [2, §8.3]. We now give that implementation.

4 The classical Boolean/CRT predicates

Use an explicitly encoded 3-CNF formula of length L , with clauses of length at most three. Reject malformed encodings, remove duplicate literals, delete tautological clauses and relabel the variable identifiers actually occurring. An empty clause is a fixed NO case; no clauses is a fixed YES case. Binary names or declarations of unused variables are never expanded. Otherwise there are $n \geq 1$ variables, every clause uses $1 \leq t \leq 3$ distinct variables, and the numbers of variables and clauses are $O(L)$.

Choose the first n primes starting at 3, denoted $p_1, \dots, p_n$. For each variable use a bank with modulus p_i and allowed set $\{0, 1\}$. A clause with variable set I uses modulus $q_I = \prod_{i \in I} p_i$ and the CRT representatives of its satisfying local Boolean assignments. It has $2^{|I|} - 1 \leq 7$ allowed residues, a nonempty proper set. They can be obtained by inspecting at most eight assignments. An integer k satisfies all bank predicates exactly when its variable residues describe a satisfying assignment. Conversely CRT lifts every satisfying assignment to some $k \geq 0$ modulo $M = \prod_i p_i$. This proves both directions without finding an assignment.

For exact attribution, Stockmeyer–Meyer [3, Theorem 6.1, pp. 7–8] use variable-invalid residues and one failed Boolean residue per clause, with the same prime and three-prime products; see

also [4, Fact 3]. If V_i means $k \bmod p_i \notin \{0, 1\}$ and F_C is the unique falsifying Boolean residue for clause C , their bad language is

$$\bigcup_i V_i \cup \bigcup_C F_C.$$

Our complete allowed-predicate intersection is exactly its complement. A local clause complement also contains non-Boolean residues, so it is not just F_C ; those additional residues are already in $\bigcup_i V_i$. This supplies the exact union/complement qualification. Neither this CRT encoding nor avoidance of expansion of M is claimed as new.

Only individual primes and products of at most three primes are enumerated. For an explicit polynomial bound, writing p_j^* for the j th prime including 2, Dusart [7, Eq. (4.2)] gives $p_j^* \leq ej \log j$ for $j \geq 2$. Thus

$$U := \max_i p_i = p_{n+1}^* \leq 3(n+1)^2.$$

A sieve up to this bound is polynomial; the $j = 2$ endpoint also satisfies it. Every local modulus is at most U^3 . The global product M supplies only a correctness witness: the constructor never expands it as a graph or simulates that many root firings.

5 A simple conservative realization

Index the variable and clause banks by $b = 1, \dots, H$, and write

$$Q = \sum_b q_b, \quad P = \max_b q_b, \quad K_b = \frac{q_b(q_b - 1)}{2} - |A_b|, \quad B = \sum_b (K_b + q_b), \quad S(k) = \sum_b S_{A_b}(k).$$

There are Q *distinct* counters: for each bank and offset a , create c_a separate vertices initially holding a chips. Equal offsets do not identify vertices. Add a root r and $P - 1$ returns $t_1, \dots, t_{P-1}$. The complete arc list is

$$\begin{array}{ll} r \rightarrow v & \text{for every counter } v, \\ v \rightarrow r, \quad v \rightarrow t_j \quad (1 \leq j < q_b) & \text{for a counter of bank } b, \\ t_j \rightarrow r & (1 \leq j < P). \end{array}$$

Hence $d_r = Q$, $d_v = q_b$ and $d_{t_j} = 1$. The arc types are disjoint, every ordered pair occurs at most once, and there are no loops. Root/counter antiparallel pairs are permitted in this first construction. A modulus- P bank has P actual counters, any of which reaches every return. The root reaches every counter, and every other vertex reaches the root; thus the graph is strongly connected. Deleting r leaves only counter-to-return arcs, an acyclic graph with no two-arc path. The exact counts are

$$N = Q + P, \quad |E| = Q + \sum_b q_b^2 + P - 1. \quad (4)$$

Initialize returns at zero and the root at

$$R = Q + B - 1 - S(0) \geq Q - 1 \geq 0. \quad (5)$$

The inequality holds because each bank is at most its own maximum. The total number of chips is exactly $C = Q + B - 1$, and every firing conserves it. The returns are ordinary degree-one vertices that return their chips to r ; they never absorb or delete them.

6 Legal checkpoints and the SAT equivalence

Analyze one legal order only for the proof: at a checkpoint all nonroot vertices are stable and the returns empty. If r is legal, fire it once, then process all counters, then all returns, before considering r again. A stable counter gains exactly one chip from that root firing and has no other incoming arc; it fires at most once, precisely when it reaches q_b . After counters have been processed, each return legally fires once per chip it holds. These finite phases restore every emitted chip to the root, and no counter is reactivated before another root firing. Delaying a legal root while these other legal vertices fire is permitted in ordinary asynchronous chip-firing. No sink rule has been substituted.

Inductively, at checkpoint k , after k root firings, each offset- a counter holds $(a + k) \bmod q_b$, all returns are zero, and the root holds

$$C - S(k) = Q + B - 1 - S(k).$$

Using integral storage and $S(k) \leq B$ gives the exact equivalences

$$r \text{ stable} \iff S(k) > B - 1 \iff S(k) = B \iff \text{every bank accepts } k. \quad (6)$$

A bank misses its maximum by exactly q_b when its predicate is false. These positive deficits cannot cancel, proving the last equivalence.

If the formula is satisfiable, CRT provides a finite successful $k \geq 0$. The canonical legal process either stops earlier, already proving halting, or reaches that checkpoint and is stable there. If it is unsatisfiable, each checkpoint permits a root firing followed by finite legal nonroot phases. Their concatenation is an infinite legal word. Lemma 1 rules out every terminating legal word, regardless of schedule and without fairness. We have therefore proved

$$\text{3-CNF is satisfiable} \iff \text{the constructed ordinary configuration halts.}$$

The canonical order and possibly enormous stopping time belong to the proof, not to the polynomial reduction algorithm.

7 Bounds, totality and the oriented construction

We have $H = O(L)$, $q_b \leq U^3$, $P \leq U^3$, and $Q \leq HP$. Equation (4) gives $N = O(HP)$ and $|E| = O(HP^2)$; an explicit adjacency matrix of N^2 bits is polynomial as well. Offset scans and CRT calculations on at most three primes take polynomial time and bits. Since $B_b = q_b(q_b + 1)/2 - |A_b|$,

$$0 \leq C = Q + B - 1 \leq Q + \sum_b \frac{q_b(q_b + 1)}{2}.$$

The *numerical* total chips and every coordinate are polynomial in L , so unary chip encoding also retains hardness. There is no expansion of a binary exponential arc multiplicity. As a consistency check, the prior stable-capacity bound $C \leq |E| - N$ for halting inputs [1, p. 2] does not trivialize the construction: here for every nontrivial input

$$(|E| - N) - C = \sum_b \left(\frac{q_b(q_b - 3)}{2} + |A_b| \right) \geq 0,$$

because $q_b \geq 3$. This bound is not being used as a SAT decision rule.

For malformed formulas or empty clauses use a directed two-cycle with configuration $(1, 0)$, a NO instance with a circulating chip. For the empty conjunction use the same graph with $(0, 0)$, a stable YES instance. These cases make the reduction total on encoded strings and preserve all main graph restrictions, including root deletion. For nontrivial formulas the construction above applies.

For the oriented version use P returns, omit every counter-to-root arc, and instead give a modulus- q_b counter the q_b arcs $v \rightarrow t_1, \dots, v \rightarrow t_{q_b}$. Keep $r \rightarrow v$ and all $t_j \rightarrow r$. Every arc follows the three-part order root-counter-return-root, so no opposite pair, loop or repeated ordered pair occurs. Every counter reaches r through a return, and a largest-bank counter reaches all P returns; strong connectivity follows. Root deletion again leaves only counter-to-return arcs. Its exact counts are

$$N' = 1 + Q + P = N + 1, \quad |E'| = Q + \sum_b q_b^2 + P = |E| + 1.$$

All outdegrees, counter offsets and root initial mass remain the same. Each ready counter now emits its q_b chips to q_b distinct returns. Legally emptying those returns sends exactly q_b chips back to the root, so the finite-phase proof, residue induction, conservation and (6) all hold with no missing direct output. Thus both SAT implications and all numerical and bit bounds remain valid. For preprocessing use the directed three-cycle $r \rightarrow s \rightarrow t \rightarrow r$, with $(0, 0, 0)$ for YES and $(1, 0, 0)$ for NO. Its deletion graph has one arc and its one-chip play is periodic. This proves the entire oriented case, including trivial inputs. Together with membership, Theorem 1 follows.

8 Priority limits and verification

The bounded source audit found exact antecedents for the bank priming and CRT encoding, which are explicitly credited above. It did not establish a source statement or proved corollary containing the entire restricted theorem. That is not evidence of worldwide novelty. Priority of the full result, the conservative simple realization, the oriented realization and any remaining method contribution remains unknown. No claim that the question was globally open immediately before this work is made.

Several nearby interfaces must be kept separate. Generalized rotor reachability [8] has branching arc states and a specified target; ordinary chip-firing has fixed outputs and asks for any stable endpoint. Bond-Levine’s sandpiling [9, Definition 6.5, Corollary 6.6] preserves halting on *all* inputs, not automatically one reset-state input. Hardness for containing weighted-net classes does not automatically restrict to one consuming transition per place, degree thresholds, unit outputs, conservation and a supplied marking. Esparza’s sole-author survey [10] does not remove those hypotheses. The full Mayr-Weihmann construction and complete Hujsa-Devillers weighted-net proofs were unavailable to this audit; their possible additional implications remain unknown. Optimization over added chip distributions, infinite backgrounds and binary arc expansion are also different decision interfaces. These statements record exact transfer gaps, rather than exclude every possible earlier derivation.

The accompanying portable package contains this source, bibliographic hashes and read scopes, frozen authored history, new standard-library exact finite controls and a deterministic archive rebuilder. It distributes no third-party PDFs or copied source bodies. New diagnostics check bank identities and the two exact antecedent substitutions, legal conservative main/oriented graphs, small arbitrary schedules, certificate slack, and deliberately damaged variants. They supplement the general proof, do not certify the literature, and are not replays of historical omitted programs. Source editions and selective read scopes are distinguished explicitly in the source ledger; the actual

journal and arXiv editions of [8] have different bytes.

AI tools were used extensively in solving, checking, adversarial review, source comparison, programming and writing. The author is responsible for the claims. This manuscript is unrefereed, has not received external human peer review or journal acceptance, and is not formally certified in a proof assistant. No individual was contacted. The paper and authored verification materials are licensed under Creative Commons Attribution 4.0 International.

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