A characteristic-two counterexample to dévissage for local deformation functors

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Abstract

We give a negative answer to the unrestricted fibre-product question for local deformation functors posed by Byszewski. For the weakly ramified action of the additive group of $\mathbb{F}_4$ given by $t \mapsto t/(1+at)$, and its subgroup $\mathbb{F}_2$, restriction–induction cannot be formally smooth over any prorepresentable recipient, regardless of its tangent dimension. The proof combines the known involution deformation pathology with an exact calculation showing that restriction has injective tangent map. The variant assuming prorepresentability of the subgroup and quotient deformation functors is untouched. The involution pathology and additive action are established prior ingredients; the contribution is their use to obstruct the proposed factorization.

1 The unrestricted question and the counterexample

Let k be a perfect field of characteristic p and $\mathcal{C}$ the category of local Artinian $W(k)$-algebras with identified residue field k. For a faithful local action $\rho : H \to \mathrm{Aut}_k\, k[[t]]$, $D_H(A)$ is the set of lifts to $\mathrm{Aut}_A\, A[[t]]$, modulo coordinate conjugation reducing to the identity. This is the unframed category of [1, 3, 4]; nilpotent constant terms are permitted. The residual action gives a unique point over k. For $N \triangleleft G$ and $Q = G/N$, restriction and induction give a natural map

$$(\mathrm{res}, \mathrm{ind}) : D_G \longrightarrow D_N^Q \times D_Q,$$

where D_N^Q is the actual fixed subfunctor of D_N. A prorepresentable functor has the form

$$h_R(A) = \mathrm{Hom}_{W(k),\mathrm{loc}}(R, A)$$

for a complete Noetherian local $W(k)$-algebra with residue k.

The question in [1, p. 3039], [2, Question 6.4.4], and [3, Question 4.4] asks for a prorepresentable F, natural maps $D_N^Q \to F \leftarrow D_Q$, and a formally smooth restriction–induction map into the fibre product. The refined formulation prescribes $TF = H^1(Q, \Theta^\sharp/\Theta^N)$, with $\Theta = \mathrm{Der}_k\, k[[t]]$ and $\Theta^\sharp = \mathrm{Der}_k(k[[t]]^N)$. Neither question assumes D_N or D_Q prorepresentable. Those extra hypotheses are stated separately in [1, Theorem 3] and [3, Remark 4.5, Theorem 4.9].

Theorem 1. *Put $k = \mathbb{F}_4 = \mathbb{F}_2[u]/(u^2+u+1)$, $G = (k, +)$, $N = (\mathbb{F}_2, +)$, and $\rho(a)(t) = t/(1+at)$. There are no prorepresentable F and natural maps $f : D_N^Q \to F$, $g : D_Q \to F$ for which restriction–induction factors as a formally smooth map*

$$D_G \longrightarrow D_N^Q \times_F D_Q.$$

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This remains impossible for every tangent dimension of F; in particular it excludes the prescribed one-dimensional tangent space in this example.

The action is a standard weakly ramified additive action, not a new construction. Direct substitution gives $\rho(a)\rho(b) = \rho(a+b)$, and the t^2 coefficient distinguishes a. Every nonidentity element has conductor one. The proof below uses actual deformation classes, without replacing either component functor by a versal hull.

2 A quotient tangent invisible to every recipient

The N-quotient parameter and induced Q-action are

$$y = \frac{t^2}{1+t}, \qquad k[[t]]^N = k[[y]], \qquad \sigma(y) = \rho(u)(y) = \frac{y}{1+y}.$$

Here $t^2+yt+y = 0$, $\rho(1)(t) = t+y$, and every series uniquely has the form $A(y)+tB(y)$: the leading terms of y^m and ty^m have even and odd t-orders, and successive leading-term subtraction converges. Invariance forces $yB(y) = 0$. The same basis argument for Laurent series gives $k((t))^N = k((y))$. Finally $u(u+1) = 1$ gives

$$\rho(u)(y) = \frac{t^2}{(1+ut)(1+(u+1)t)} = \frac{t^2}{1+t+t^2} = \frac{y}{1+y}.$$

Thus D_Q is the weakly ramified involution deformation functor.

Lemma 2. *There is a nonzero $v \in D_Q(k[\varepsilon]/\varepsilon^2)$ whose image under every natural map $D_Q \to h_R$ is the residue basepoint.*

Proof. Let $B = k[\varepsilon]/\varepsilon^3$, $A = k[\varepsilon]/\varepsilon^2$, and let $j : B \to B$ fix k and send ε to $\varepsilon+\varepsilon^2$. These are objects and maps of $\mathcal{C}$ via $W(k) \to k$. Define

$$\sigma_\varepsilon(y) = \frac{y+\varepsilon}{1+y}, \qquad M_\varepsilon = \begin{pmatrix} 1 & \varepsilon \\ 1 & 1 \end{pmatrix}.$$

Its constant coefficient is nilpotent and its linear coefficient is a unit, so substitution defines a continuous automorphism. Its matrix square is $(1+\varepsilon)I$, so it is an involution lifting σ. Write $x = [\sigma_\varepsilon] \in D_Q(B)$ and v for its reduction to A.

The class x is fixed by j. The identity-reducing coordinate change $\chi(y) = (1+\varepsilon)y+\varepsilon$, represented by $C = \left(\begin{smallmatrix} 1+\varepsilon & \varepsilon \\ 0 & 1 \end{smallmatrix}\right)$, satisfies

$$M_{\varepsilon+\varepsilon^2}C = (1+\varepsilon)CM_\varepsilon$$

in B. Scalar matrices do not change the fractional-linear substitution, so this gives the required conjugacy. Function composition reverses the source's ring-substitution order; the equivalence is valid in either conjugating direction. This is the known calculation of [4, Proposition 4.1], reproduced here for completeness.

For a natural map $\eta : D_Q \to h_R$, write $\psi : R \to B$ for $\eta_B(x)$. Naturality gives $j \circ \psi = \psi$. Since

$$B^j = k + k\varepsilon^2,$$

the composite $R \to B \to A$ lands in $k \subset A$. It is consequently the residue map followed by $k \to A$, hence $\eta_A(v)$ is the basepoint. This argument applies also to mixed-characteristic R and arbitrary tangent dimension, with no linearity assumption on η.

The class v is nonzero. Every identity-reducing coordinate change over A has the form $y+\varepsilon h(y)$ with arbitrary $h \in k[[y]]$, including its constant coefficient. Its first-order conjugacy perturbation is, up to sign, $h(\sigma(y)) - \sigma'(y)h(y)$, whose constant coefficient is zero. But $\sigma_\varepsilon - \sigma = \varepsilon/(1+y)$ has constant perturbation coefficient one. Thus no allowed conjugator trivializes v. □

The conclusion of Lemma 2 is also an elementary consequence of the previously published failure to preserve injections in [4, Proposition 6.1, p. 899]. There the nonzero involution tangent and the basepoint become equal after the injection $k[e]/e^2 \hookrightarrow k[w]/w^3$, $e \mapsto w^2$. Every h_R preserves injections of target rings. Naturality therefore makes their images equal already over $k[e]/e^2$. We claim no new recipient-collapse phenomenon or new involution pathology; the direct cubic-ring proof fixes the particular tangent class used here.

3 The restriction tangent is injective

Over k, every algebraic derivation of $k[[t]]$ is a coefficient derivation: perfectness in characteristic two gives $h = a(t)^2 + tb(t)^2$, so $d(h) = d(t)b(t)^2 = d(t)h'(t)$. Thus $\Theta = k[[t]]\partial_t$, and these derivations are continuous. There is no additional discontinuous tangent module in this example.

Restriction to the N-fixed ring identifies

$$\Theta^N \cong M := yk[[y]]\partial_y.$$

Indeed the unique extension of $b(y)\partial_y$ to the separable field $k((t))$ has coefficient

$$a(t) = \frac{b(y)}{dy/dt} = b(y)\frac{(1+t)^2}{t^2}, \qquad \frac{dy}{dt} = \frac{t^2}{(1+t)^2}.$$

Uniqueness makes it N-invariant. It preserves $k[[t]]$ precisely when $2\operatorname{ord}_y b - 2 \geq 0$, namely $b \in yk[[y]]$. Conversely an invariant derivation restricts and is recovered by this formula. The field polynomial has derivative $y \neq 0$, so separability and uniqueness hold.

Conjugation by the quotient involution acts on coefficient functions by

$$T(f)(y) = (1+y)^2 f\left(\frac{y}{1+y}\right), \qquad T^2 = 1.$$

All group elements are involutions, so the inverse-conjugation convention gives the same operator. Put $z = y^2/(1+y)$. The preceding fixed-ring basis argument gives $k((y))^Q = k((z))$, and nonzero invariant Laurent series have even y-valuation. For $f \in yk[[y]]$, $Tf = f$ is equivalent to invariance of f/y^2. Its valuation is at least -1, hence by evenness at least zero. It follows that

$$M^Q = y^2 k[[z]]\partial_y, \qquad (T-1)(yh(z)) = y^2 h(z) \quad (h \in k[[z]]).$$

The reverse containment of $(T-1)M$ in M^Q follows from $(T-1)^2 = 0$. The cyclic cocycle formula therefore gives

$$H^1(Q, \Theta^N) = \frac{\ker(1+T)}{(T-1)M} = 0.$$

These are identities on complete formal series, not finite-jet calculations.

The first-order lifting calculation identifies $TD_H = H^1(H, \Theta)$, and restriction with cohomological restriction [3, Proposition 2.4]: write a lift as $(\mathrm{id} + \varepsilon c(g))\rho(g)$; group relations give a cocycle, and coordinate conjugation changes it by a coboundary. The vanishing above makes restriction injective. Explicitly, for a cocycle whose restriction is zero in cohomology, subtract a coboundary

so it vanishes on N. The cocycle equation and commutativity of G make its value at u N-invariant; the relation $u+u=0$ gives $(1+T)c(u)=0$. Write $c(u)=(T-1)m$ with $m\in\Theta^N$ and subtract this coboundary too. It preserves the zero N-values and kills u, hence kills the entire cocycle. Thus

$$\ker(TD_G \longrightarrow TD_N)=0.$$

Since $D_N^Q(A)\subset D_N(A)$ literally, restriction into the actual fixed subfunctor also has zero tangent kernel.

For the prescribed space, $\Theta^\sharp = k[[y]]\partial_y$ and $\Theta^\sharp/\Theta^N\cong k$ has trivial Q-action (constant coefficients are preserved by T). Hence $H^1(Q,\Theta^\sharp/\Theta^N)=k$, as asserted in the theorem. Its dimension is not needed for the contradiction.

4 Failure of formal smoothness and scope

Proof of Theorem 1. Suppose such F, f and g exist; identify F naturally with h_R. At $A=k[\varepsilon]/\varepsilon^2$, naturality from k sends the residual basepoint $0_N\in D_N^Q(A)$ under f to the residue basepoint of $F(A)$. Lemma 2 gives the same image for $g_A(v)$, with $v\neq 0$. Therefore $(0_N,v)$ belongs to $(D_N^Q\times_F D_Q)(A)$. Its reduction is the unique residual point. The surjection $A\to k$ is small: its kernel is annihilated by the maximal ideal of A. Formal smoothness would lift this pair to $w\in D_G(A)$ with

$$\mathrm{res}(w)=0_N,\qquad \mathrm{ind}(w)=v.$$

Restriction injectivity forces $w=0_G$. Naturality of induction then gives $\mathrm{ind}(w)=0_Q$, contradicting $v\neq 0$. □

The cubic ring is used only to constrain detection of a first-order tangent by an arbitrary prorepresentable recipient. The obstruction to smoothness already occurs on $A\to k$. No extension of f from D_N^Q to D_N is assumed, and no cohomological tangent formula for f or g is prescribed. Nor is universality of a Klein-four deformation ring used.

This answers the unrestricted original question negatively and does not contradict the conditional obstruction-space theorems cited above. The involution component is in the exceptional nonprorepresentable case. The variant imposing prorepresentability of D_N and D_Q remains unaddressed; changing to coordinate automorphisms fixing the zero section would likewise change the category and exclude the crucial nilpotent translation. The introductory and refined formulations are one question, not separate discoveries.

Priority, verification and review status. The known additive action, involution conjugacy, failure to preserve injections, and tangent-cohomology framework are credited above. The supplement records independent formal-series and functor audits, finite exact checks and a bounded priority search. No matching prior negative factorization was located in inspected primary bodies; this does not prove exhaustive originality or that the question remained open until 2026. Finite computations support explicit identities; the unrestricted formal series and all-natural-map statements are proved in the text, without a formal proof-assistant certificate.

AI tools were used extensively in solving, verifying, auditing sources, and preparing this paper and supporting materials. This is an unrefereed research note. Independent adversarial reviews are AI reviews; conventional human peer review and journal acceptance are not claimed.

References

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[4] J. Byszewski and G. Cornelissen, *Which weakly ramified group actions admit a universal formal deformation?*, Ann. Inst. Fourier **59** (2009), 877–902. doi:10.5802/aif.2450, Propositions 4.1 and 6.1.