A two-pole proof of the centered pedal-area invariant

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Abstract

We give a Jacobi elliptic-function proof that the signed area of a periodic elliptic-billiard orbit times the signed area of its pedal polygon from the common center is constant for least periods odd or divisible by four. The same scoped theorem appears in Ferudun's contemporaneous preprint; this is an attributed alternative proof, with no theorem, formula, method, priority or independent-discovery claim. The proof identifies the pedal area with a shifted cyclic dn sum by canceling apparent double poles between incident area terms. A two-pole divisor then gives the invariant. Primitive star orbits are included. We make the necessary least-period convention explicit and give an exact repeated-orbit counterexample to an unrestricted list-length formulation. Verification materials accompany this AI-assisted, unrefereed research note.

1 Statement, scope and attribution

Fix $a > b > 0$ and $0 < \lambda < b^2$, and consider

$$E : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad C : \frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1. \tag{1}$$

Thus C is a proper, strictly nested confocal elliptical caustic. For an ordered closed polygon $V = (V_j)_{j \in \mathbb{Z}/N\mathbb{Z}}$ set

$$\mathcal{A}(V) = \frac{1}{2} \sum_{j=0}^{N-1} \det(V_j, V_{j+1}).$$

Let P be a physical billiard orbit on E tangent to C. Its ordered center-pedal polygon consists of the perpendicular feet Q_j from $O = (0, 0)$ to the *complete original sidelines* $P_j P_{j+1}$. Write $A = \mathcal{A}(P)$ and $B = \mathcal{A}(Q)$.

Theorem 1. *If the least physical period $N \geq 3$ is odd or divisible by four, then AB is independent of phase in the fixed-caustic family (1). Every admitted coprime winding number, including primitive stars, and either orientation are allowed.*

The observable, parity and center restriction are the experimentally reported invariant $k_{203,b}$ of Reznik, Garcia and Koiller [1, 2, Table 3]. Their area convention is signed, and their table does not literally specify least period. We state that qualification explicitly: Section 6 shows that a version based only on the length of a repeated vertex list is false. No hyperbolic, collapsed or nonconfocal caustic, moving pedal point, clipped segment projection, or unweighted filled-region area of a star is covered.

Ferudun [3, Theorem 1] proves exactly this qualified theorem, including its star and repeated-list conventions. We credit that result prominently and compare the trace formulas in Section 7. The present note records another proof organization and reproducibility materials; it does not claim a new resolution of a presently open problem. Stachel's canonical parametrization and the established meromorphic methods of [6, 7] are prior mathematics.

2 Canonical coordinates and the two areas

Put

$$\alpha = \sqrt{a^2 - \lambda}, \quad \beta = \sqrt{b^2 - \lambda}, \quad k = \frac{\sqrt{\alpha^2 - \beta^2}}{\alpha}, \quad k' = \frac{\beta}{\alpha}.$$

All Jacobi functions below have modulus $k \in (0, 1)$. Let $K = K(k)$ and $K' = K(k')$ be the complete elliptic integrals of the first kind. Stachel [4, Theorem 4.3 and (4.9)] gives

$$P(w) = (-a \operatorname{sn} w, b \operatorname{cn} w), \qquad a = \frac{\alpha \operatorname{dn} v}{\operatorname{cn} v}, \quad b = \frac{\beta}{\operatorname{cn} v}, \quad \delta = 2v = \frac{4K\tau}{N}, \tag{2}$$

where $0 < v < K$, $0 < \tau < N/2$ and $\gcd(N, \tau) = 1$, after choosing the orientation. The vertices are $P(w + j\delta)$. Conversely these parameters describe every proper pair (1) admitting such a periodic family: $\operatorname{cn} v = \beta/b$ has a unique solution in $(0, K)$, and confocality gives the other relation in (2). The real map P parametrizes E once modulo $4K$, so the coprimality condition is precisely the least-period condition. It does not restrict the winding number to one.

The contact phase is useful to verify directly. The line through $P(u - v)$ and $P(u + v)$ is

$$n(u) \cdot X = 1, \qquad n(u) = \left(-\frac{\operatorname{sn} u}{\alpha}, \frac{\operatorname{cn} u}{\beta} \right). \tag{3}$$

It is tangent to C at $(-\alpha \operatorname{sn} u, \beta \operatorname{cn} u)$. Substitution of either endpoint using the addition formulas reduces the numerator to $\operatorname{sn}^2 u \operatorname{dn}^2 v + \operatorname{cn}^2 u = 1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v$, the addition denominator. The actual Euclidean center foot is therefore

$$Q(u) = \frac{n(u)}{n(u) \cdot n(u)} = \left(-\frac{\alpha k'^2 \operatorname{sn} u}{\operatorname{dn}^2 u}, \frac{\beta \operatorname{cn} u}{\operatorname{dn}^2 u} \right). \tag{4}$$

For real u, $n(u) \cdot n(u) = \operatorname{dn}^2 u / \beta^2 > 0$. In complex continuation the dot product is bilinear, with no conjugation.

Define

$$S(w) = \sum_{j=0}^{N-1} \operatorname{dn}(w + j\delta), \qquad T(u) = \frac{1}{2} \sum_{j=0}^{N-1} \det(Q(u + j\delta), Q(u + (j+1)\delta)). \tag{5}$$

Then $B(w) = T(w + v)$. The addition identities give

$$\det(P(u - v), P(u + v)) = \frac{2ab \operatorname{sn} v \operatorname{cn} v \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 v \operatorname{sn}^2 u},$$
$$\operatorname{dn}(u - v) + \operatorname{dn}(u + v) = \frac{2 \operatorname{dn} v \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 v \operatorname{sn}^2 u}.$$

After the shoelace factor $1/2$, summing counts each vertex dn term twice. Hence

$$A(w) = C_A S(w), \qquad C_A = \frac{ab \operatorname{sn} v \operatorname{cn} v}{\operatorname{dn} v} > 0. \tag{6}$$

These are signed determinant identities; no simplicity assumption on the polygon has been made.

3 The divisor of the cyclic delta-amplitude

Set

$$m = \begin{cases} N & N \text{ odd}, \\ N/2 & N \text{ even}, \end{cases} \qquad \ell = \frac{2K}{m}, \qquad p = \mathrm{i}K'. \tag{7}$$

The order of δ modulo $2K$ is m. Its N residues have multiplicity N/m, and the generating integer δ/ℓ is coprime to m. Reindexing and the $2K$ period of dn show that S is even and ℓ-periodic. It has period $4p$ and antiperiod $2p$. We use the standard poles, zeros and shifts in [5, §22.4]; in particular $\operatorname{dn}(z+2K) = \operatorname{dn} z$ and $\operatorname{dn}(z+2p) = -\operatorname{dn} z$.

Lemma 1. *On $X = \mathbb{C}/(\ell\mathbb{Z} + 4p\mathbb{Z})$,*

$$\operatorname{div}(S) = [p + \ell/2] + [3p + \ell/2] - [p] - [3p].$$

Consequently $S(w)S(w+\ell/2)$ is a positive real constant for real w.

Proof. The poles of dn are simple and occur at $2rK + (2s+1)p$. On X the only possible poles of S are p and $3p$. At p exactly N/m summands have a pole; their arguments differ by real periods, so their residues agree and do not cancel. Indeed $\operatorname{dn}(p+z) = -\mathrm{i}/z + O(z)$, so the residue is $-\mathrm{i}N/m$. The antiperiod gives the opposite residue at $3p$.

At $z = p + \ell/2$, which is not a pole, the equality $-z + \ell + 2p = z$ combines evenness and the antiperiod to give $S(z) = -S(z)$. The same holds at $3p + \ell/2$. These two distinct zeros exhaust the zero divisor, since an elliptic function with two simple poles has total zero order two. Both zeros are simple. The divisor of $S(w)S(w+\ell/2)$ is therefore zero; compactness makes the product constant. On the real axis every summand dn is positive, so the constant is positive. □

4 Cancellation in the pedal-area trace

Lemma 2. *For a real constant C_T, one has $T(u) = C_T S(u+K)$. In fact*

$$C_T = \alpha^2 \frac{\operatorname{sn}\delta(\operatorname{dn}\delta + k'^2)}{(1+\operatorname{dn}\delta)^2} > 0. \tag{8}$$

Proof. At a common pole of $\operatorname{sn}, \operatorname{cn}, \operatorname{dn}$, the numerators in (4) have order one and dn^2 order two; Q is holomorphic there and vanishes. Its remaining possible poles are the simple zeros of dn, at $r = K + p$ and their translates. They have order at most two for Q. The shift identities and parity give

$$Q(2r - z) = Q(z). \tag{9}$$

Thus $Q(r+\varepsilon)$ has only even Laurent powers, with no ε^{-1} term.

At a phase $u + j\delta = r + \varepsilon$, the two incident summands of the ordered area trace combine as

$$\frac{1}{2}\det\big(Q(r+\varepsilon), Q(r+\delta+\varepsilon) - Q(r-\delta+\varepsilon)\big). \tag{10}$$

The neighbors are holomorphic because $0 < \delta < 2K$ is not a real pole-to-pole shift. Their difference vanishes at $\varepsilon = 0$ by (9). Its order at least one reduces the apparent double pole of (10) to at most a simple pole. If N is even, the $N/m = 2$ singular indices are separated by $m \geq 2$; no edge has two singular endpoints, so the incident grouping remains valid without counting an edge twice. This includes $N = 4$, when the regular neighbors vanish. For $N = 3$ there is only one singular index.

From (4),

$$Q(u+2K) = -Q(u), \qquad Q(u+2p) = \operatorname{diag}(1,-1)Q(u).$$

Determinants make T periodic under $2K$ and antiperiodic under $2p$. The closed cyclic sum is also δ-periodic, hence ℓ-periodic. It is meromorphic on X, with only possible simple poles at r and $r+2p$. These are exactly the poles of $S(u+K)$, modulo the real period $2K$. Match one residue by a scalar C_T; the antiperiod matches the other. The difference is a holomorphic function on the compact torus, hence constant, and its antiperiod forces it to be zero.

For completeness we evaluate that scalar. The quarter-period formulas give

$$Q(r+x) = \frac{\alpha}{k}\frac{\operatorname{cn} x}{\operatorname{sn}^2 x}(\operatorname{dn} x, \mathrm{i}), \qquad U = \frac{\alpha}{k}(1,\mathrm{i}), \qquad Q(r+\varepsilon) = U\varepsilon^{-2} + O(1).$$

The neighbor difference in (10) is $2\varepsilon Q'(r+\delta) + O(\varepsilon^3)$; the residue of T is $(N/m)\det(U, Q'(r+\delta))$. The residue of $S(u+K)$ is $-\mathrm{i}N/m$. With $q_z = Q_x + \mathrm{i}Q_y$, the transformed formula simplifies to

$$q_z(r+x) = -\frac{\alpha k \operatorname{cn} x}{1+\operatorname{dn} x}, \qquad q_z'(r+x) = \frac{\alpha k \operatorname{sn} x(\operatorname{dn} x + k'^2)}{(1+\operatorname{dn} x)^2}.$$

Therefore $C_T = \mathrm{i}\det(U, Q'(r+\delta)) = (\alpha/k)q_z'(r+\delta)$, which is (8). Since $0 < \delta < 2K$, its real factors give $C_T > 0$. No division by a polygon area was used. □

5 Parity and conclusion

Equations (6) and Lemma 2 give

$$A(w)B(w) = C_A C_T S(w)S(w+K+v). \tag{11}$$

If N is odd, $(K+v)/\ell = N/2 + \tau$ is a half-integer. If N is even it equals $(m+\tau)/2$. For $4 \mid N$, m is even and τ odd, so this too is a half-integer. In both target cases $K + v \equiv \ell/2 \pmod{\ell}$, and Lemma 1 proves Theorem 1.

For excluded primitive $N \equiv 2 \pmod 4$, m and τ are odd, so the shift is an integer multiple of ℓ. The product is $C_A C_T S(w)^2$ and is genuinely nonconstant: S has poles and both coefficients are positive. Constancy on a real interval would contradict the meromorphic identity theorem. Reversal negates A and B and preserves their product. In the chosen orientation both areas are positive, by (6) and (8). Taking absolute values of these signed shoelace areas also preserves the product; this says nothing about unweighted filled-in areas of stars.

For example the least-four caustic has $\lambda = a^2b^2/(a^2+b^2)$. At its axial diamond, $A = 2ab$ and $B = 4a^3b^3/(a^2+b^2)^2$, hence the invariant is $8\lambda^2$. This value and low-period results are also in the overlapping literature [3, 8].

6 Why a repeated list does not change the period

An exact boundary check uses $a = 2$, $b = 1$ and $\lambda = 4/9$. The following two positive physical six-period orbits have that caustic:

$$\begin{aligned} H = ((2,0), (4/3, \sqrt{5}/3), (-4/3, \sqrt{5}/3),\\ (-2,0), (-4/3, -\sqrt{5}/3), (4/3, -\sqrt{5}/3)),\\ J = ((0,1), (-4\sqrt{2}/3, 1/3), (-4\sqrt{2}/3, -1/3),\\ (0,-1), (4\sqrt{2}/3, -1/3), (4\sqrt{2}/3, 1/3)). \end{aligned}$$

Direct substitution verifies ellipse membership, tangent discriminants and forward specular reflection at every vertex; all six vertices are distinct. Direct perpendicular projection and shoelace summation give

	A	B	AB
H	$20\sqrt{5}/9$	$140\sqrt{5}/81$	$14000/729$
J	$32\sqrt{2}/9$	$224\sqrt{2}/81$	$14336/729$

Double traversal multiplies each area by two and its product by four. The two twelve-entry products differ by $448/243$, while their least period stays six. Thus a list length divisible by four is insufficient. Ferudun [3, Section 6] already gives the same obstruction with another exact physical example; no new boundary phenomenon is claimed here.

7 Comparison and the documented chronology

Let $W(u) = a^2 - (a^2 - b^2)\,\mathrm{sn}^2\,u$. Ferudun's reflection-coordinate telescope expresses B as a constant times $\sum_j \mathrm{dn}(w + j\delta)/W(w + j\delta)$ [3, Lemma 3]. The ordinary Jacobi addition and quarter-period formulas give

$$\mathrm{dn}(u + K - v) + \mathrm{dn}(u + K + v) = \frac{2ab\,\mathrm{dn}\,u}{W(u)}.$$

For instance, its addition denominator satisfies

$$\mathrm{dn}^2\,u - k^2\,\mathrm{cn}^2\,u\,\mathrm{sn}^2\,v = \mathrm{cn}^2\,v\,W(u)/\alpha^2,$$

and $k'\,\mathrm{dn}(v)\alpha^2/\,\mathrm{cn}^2\,v = ab$. Summing around the closed orbit reindexes the two shifted terms, giving

$$\sum_j \frac{\mathrm{dn}(w + j\delta)}{W(w + j\delta)} = \frac{1}{ab} S(w + K + v).$$

Thus our normalization, including the explicit coefficient (8), is already implied by that preprint and a standard addition identity. Our direct-foot pole comparison is a different organization, not a verified new formula or method. Ferudun's later arbitrary-point paper also uses adjacent-foot pole cancellation [8, Lemma 4.3]. Earlier meromorphic cancellation and compactness arguments are in [6, 7].

The GitHub server record for our original submission, PR204, reports creation on 1 October 2026 at 05:28:14 UTC, with the preserved proof at commit `551a524e2db6` (full identifier archived) [9]. Ferudun's centered Zenodo record reports creation on the same date at 15:39:54.821511 UTC; DataCite records registration at 15:39:55 UTC. These are contemporaneous dated records. They do not by themselves establish discovery times, earliest historical visibility, priority, dependence, independence or literal simultaneity. Our original proof materials for the neighboring $k_{203,a}$ and $k_{203,b}$ efforts were shared as author contributions, not independent reviews. The present origin-only proof does not assume an arbitrary-point invariant.

8 Verification and disclosure

The accompanying archive preserves the original submission, source and priority ledgers, independent low-period derivations, and computational checks. Independent Euclidean shooting used 63 primitive shape/winding cases and 378 phase/direction orbits; exact radical controls verify two phases each for periods three and four and the six-period counterexample. Those are finite evidence. High-precision computations are noninterval diagnostics, not a proof of the universal theorem.

The historical author/reviewer scripts reproduce their reported outputs with assertions enabled, but lose correctness enforcement under Python optimization. Separate explicit-exception derivatives preserve their valid outputs and reject genuine false mathematical identities in both modes. Current numerical diagnostics also reject nonfinite values. The archive distinguishes untouched historical evidence from corrected live verification code. The analytic proof above, rather than counts or floating-point agreement, establishes the universal result.

AI tools were used extensively in the original derivation, mathematical auditing, calculations, software and manuscript preparation. Internal adversarial AI reviews do not constitute conventional human peer review. This is an unrefereed preprint and has no proof-assistant certification. The author is responsible for its claims. No outside person was contacted on behalf of this research.

References

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