

BV generators for polynomial Weyl subalgebras

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22 October 2026

Preprint, version 1.0.0

Abstract

Let k be any field of characteristic zero and let $0 \neq h \in k[x]$. For $A_h = k\langle x, Y \rangle / (Yx - xY - h(x))$, the ordinary Hochschild cohomology, with its canonical cup product and Gerstenhaber bracket, admits a Batalin–Vilkovisky generator if and only if h is squarefree. In the squarefree case its bracket is zero. A repeated irreducible factor produces a cup-zero, noncommuting pair of degree-one classes, and a central annihilator forces every putative generator to vanish on degree one. Thus in that case no degree-minus-one bracket generator exists, even without requiring its square to be zero. The argument uses the published resolution and derivation structure of this family, and answers the concrete-family question recorded by Lopes in Oberwolfach Report 46/2023, p. 2702, Question (2).

1 Question, conventions, and statement

Throughout, cohomology means ordinary $\mathrm{HH}^\bullet(A, A)$; its product and bracket are the canonical ones. A bracket generator is a k -linear map $\Delta : \mathrm{HH}^n \rightarrow \mathrm{HH}^{n-1}$ such that, for homogeneous a, b ,

$$[a, b] = (-1)^{|a|} \Delta(a \smile b) - (-1)^{|a|} \Delta(a) \smile b - a \smile \Delta(b). \quad (1)$$

A BV generator additionally satisfies $\Delta^2 = 0$. Negative cohomology is zero, so $\Delta(1) = 0$. This is the convention of the original question [5, p. 2702]; in degree one the bracket is the commutator of derivations. Units are regarded as squarefree polynomials.

Theorem 1. *For every characteristic-zero field k and every nonzero $h \in k[x]$, the following are equivalent:*

1. *h is squarefree;*
2. *the canonical Gerstenhaber bracket on $\mathrm{HH}^\bullet(A_h)$ is zero;*
3. *$\mathrm{HH}^\bullet(A_h)$ admits a bracket generator;*
4. *$\mathrm{HH}^\bullet(A_h)$ admits a BV generator.*

When h is squarefree, $\Delta = 0$ is a BV generator. When h has a repeated irreducible factor, no bracket generator exists.

For $h \neq 0$, this presentation is the subalgebra of the first Weyl algebra $k\langle x, y \rangle / ([y, x] - 1)$ generated by x and $Y = yh(x)$. The original report uses $h(x)y$; since $yh - hy = h'$, the embedded subalgebras agree. Lopes asks whether there are BV structures when this family's Nakayama automorphism is not semisimple [5, Question (2)]. The theorem classifies all nonzero parameters, including every nonconstant parameter in that branch. Section 5 treats the literal embedded specialization $h = 0$ separately.

The structural results used below are due to Benkart–Lopes–Ondrus [1] and Lopes–Solotar [6]. Nonexistence of a generator for ordinary Hochschild Gerstenhaber cohomology is already known: Herscovich–Li prove it for $\text{FK}(3)$ [3, Proposition 4.15]. Their use of cup/bracket relations to contradict every candidate operator is an essential predecessor of the obstruction strategy here. Our contribution is the squarefree classification for this polynomial family, deduced from its established structure and the cup comparison below. We make no claim to the first no-generator example or an exhaustive historical priority determination.

2 Published structure and elementary consequences

Set $A = A_h$, $d = \gcd(h, h')$, with monic $\gcd$, and $\pi = h/d$. The PBW basis is $\{x^i Y^j : i, j \geq 0\}$. We use the following precise published imports:

- The length-two free A -bimodule resolution of [6, §3, (3.1)], with degree-one generators x, Y and degree-two generator r , has

$$\begin{aligned} \partial_2(r) &= [Y]x + Y[x] - [x]Y - x[Y] \\ &\quad - \sum_{s \geq 1} h_s \sum_{i=0}^{s-1} x^i [x] x^{s-1-i}, \quad h = \sum_s h_s x^s. \end{aligned} \quad (2)$$

Here $[x], [Y]$ denote free module generators, not commutator symbols.

- In characteristic zero, $\text{HH}^2(A) = A/dA$ and $\text{HH}^n(A) = 0$ for $n \geq 3$ [6, Proposition 3.9 and Corollary 3.11].
- For $f \in k[x]$, write $D_f(x) = 0, D_f(Y) = f$. The classes of these derivations and of ad_{ga_n} , $g \in k[x]$, $n \geq 0$, span $\text{HH}^1(A)$. In the localized ambient Weyl algebra,

$$a_n = \pi h^{n-1} y^n \quad (n \geq 1), \quad a_0 = 1/d.$$

These elements normalize A ; moreover ad_{ga_n} is inner whenever $d \mid g$. These are the derivation-classification results of [1, Theorems 5.3, 5.7, 5.13 and Proposition 5.9], explicitly recalled in [6, §6.1 and Theorem 6.2]. We also use the ambient basis $A = \bigoplus_{m \geq 0} k[x] h^m y^m$ [6, Proposition 2.1(b)].

These are established structural inputs, not results certified by the finite checks supplied with this note.

For completeness, the center and the needed innerness test are elementary. If $t = \sum_{j=0}^N t_j(x) Y^j$, $N > 0$, then the leading coefficient of $[t, x]$ is $N h t_N$, of Y -degree $N - 1$. Thus $C_A(x) = k[x]$. Commuting also with Y gives $h t' = 0$, so $Z(A) = k = \text{HH}^0(A)$. If $D_f = \text{ad}_t$, then $t \in k[x]$ and $D_f(Y) = -h t'$. Conversely every multiple of h has that form by polynomial integration. Consequently

$$[D_f] = 0 \text{ in } \text{HH}^1(A) \iff h \mid f. \quad (3)$$

We use $\text{ad}_t(z) = [t, z]$.

One can also see the stated quotient in the resolution directly. The degree-one cochain differential is

$$(\alpha, \beta) \mapsto [\beta, x] + [Y, \alpha] - \sum_s h_s \sum_{i=0}^{s-1} x^i \alpha x^{s-1-i}.$$

The triangular PBW calculation gives $[A, x] = hA$, and coefficient integration gives $[Y, A] = hA$. Modulo hA , the last sum is $h'\alpha$. Therefore the image is $hA + h'A = dA$. The ideal dA is two-sided, since $[Y, d] = hd'$ is divisible by d .

Lemma 2. *The class $c = [D_d]$ is central in the Lie algebra $\text{HH}^1(A)$. If $d \neq 1$, it is nonzero and has a linearly independent companion $[D_{d/u}]$, for every irreducible $u \mid d$.*

Proof. The centrality also follows from [6, Theorem 6.2(b)]; we give its derivation from the spanning input to specify exactly what is used. Extend D_d to the localized ambient Weyl algebra by $D_d(y) = 1/\pi$, $D_d(x) = 0$. This respects $[y, x] = 1$ and $Y = yh$. For $f = 1/\pi$, commuting y through coefficients gives

$$D_d(y^n) = \sum_{j=0}^{n-1} \binom{n}{j+1} f^{(j)} y^{n-1-j}.$$

Define $P_0 = 1$ and $P_{j+1} = \pi P'_j - (j+1)\pi' P_j$; induction gives $f^{(j)} = P_j/\pi^{j+1}$. Hence

$$D_d(ga_n) = \sum_{j=0}^{n-1} \binom{n}{j+1} g d^j P_j h^{n-1-j} y^{n-1-j} \in A \quad (n \geq 1).$$

Thus $[D_d, \text{ad}_{ga_n}] = \text{ad}_{D_d(ga_n)}$ is inner. For $n = 0$, $D_d(g/d) = 0$; also $[D_d, D_f] = 0$. The spanning input proves centrality for all of HH^1 . If $u \mid d$, then d and d/u are linearly independent polynomials of degree less than $\deg h$. Equation (3) proves the last assertions. $\square$

3 The required canonical cup product

Lemma 3. *For every derivation E of A and every $f \in k[x]$,*

$$[D_f] \smile [E] = [fE(x)] \in \text{HH}^2(A) = A/dA.$$

In particular $c \smile z = 0$ for every $z \in \text{HH}^1(A)$.

Proof. Use normalized bars, with $b[a|b] = a[b] - [ab] + [a]b$. A comparison from the small resolution to the normalized bar resolution sends the degree-one generators to $[x]$, $[Y]$, and r to

$$F_2(r) = [Y|x] - [x|Y] - \sum_{s \geq 2} h_s \sum_{i=1}^{s-1} [x^i|x] x^{s-1-i}. \quad (4)$$

Indeed, for $s \geq 2$, telescoping gives

$$b\left(\sum_{i=1}^{s-1} [x^i|x] x^{s-1-i}\right) = \sum_{i=0}^{s-1} x^i [x] x^{s-1-i} - [x^s].$$

The first two bars of (4) contribute the four outer x, Y terms and $-[Yx] + [xY] = -[h]$. Constants vanish in normalized bars; the linear term left over is $-h_1[x]$. This is precisely (2). The degree-zero

and degree-one chain conditions are immediate, so this is a comparison lifting the identity. Since the small complex is the published free resolution, pullback of the ordinary bar cup product computes the canonical product.

For arbitrary derivations D, E , its value on r is

$$D(Y)E(x) - D(x)E(Y) - \sum_{s \geq 2} h_s \sum_{i=1}^{s-1} D(x^i)E(x)x^{s-1-i}.$$

The right factor must remain on the right: $E(x)$ need not commute with x . With $D = D_f$, all polynomial correction terms and the second term vanish literally, leaving $fE(x)$. For $f = d$, this belongs to dA , proving the last assertion. $\square$

4 An obstruction for all possible generators

Lemma 4. *Let H be a nonnegatively graded Gerstenhaber algebra over k with $H^0 = k$. Suppose $0 \neq c \in H^1$ is Lie-central, $cH^1 = 0$, and $\dim_k H^1 > 1$. Every degree-minus-one bracket generator on H vanishes on H^1 . Consequently every cup-zero pair in H^1 would have zero bracket.*

Proof. Write $\lambda = \Delta|_{H^1} : H^1 \rightarrow k$. The degree-one part of (1) is

$$[a, b] = -\Delta(ab) + \lambda(a)b - \lambda(b)a.$$

Taking $a = c$ gives $\lambda(c)b = \lambda(b)c$. A b independent of c first forces $\lambda(c) = 0$. Then $c \neq 0$ forces $\lambda(b) = 0$ for every b . The displayed identity proves the consequence. No condition on $\Delta|_{H^2}$ or on Δ^2 was used. $\square$

Proof of Theorem 1. If $d = 1$, the published spanning statement makes every ad_{ga_n} inner. Thus $\text{HH}^1(A)$ is spanned by the commuting $[D_f]$. All brackets with $\text{HH}^0(A) = k$ vanish, and $\text{HH}^{\geq 2}(A) = 0$. Therefore the entire bracket is zero and $\Delta = 0$ is a BV generator.

Suppose $d \neq 1$. Choose an irreducible nonconstant $u \mid d$ over k itself, and set $g = d/u$, $a = [D_g]$. If $h = \alpha \prod_i u_i^{e_i}$ is its irreducible factorization, then $d = \prod_i u_i^{e_i-1}$ in characteristic zero, so $\pi = \alpha \prod_i u_i$. In particular $u \mid \pi$ and

$$L = \pi \frac{d'}{d} = \sum_i (e_i - 1) \frac{\pi}{u_i} u'_i \in k[x].$$

Define b directly on the presentation by $b(x) = \pi$, $b(Y) = LY$. Write $\beta = [b]$. It is a derivation because its value on the defining relation is

$$[LY, x] + [Y, \pi] - \pi h' = Lh + h\pi' - h'\pi = 0.$$

Lemma 3 gives $a \smile \beta = [g\pi] = 0$, since $g\pi = d(\pi/u)$. On generators the commutator has x -component zero and

$$[D_g, b](Y) = Lg - \pi g' = \frac{h}{u^2} u' =: q.$$

Since $u^2 \mid h$ and $u' \neq 0$, this is a nonzero polynomial. In characteristic zero, $\deg q = \deg h - \deg u - 1 < \deg h$, so (3) gives $[a, \beta] = [D_q] \neq 0$. Lemmas 2–4 now contradict every bracket generator. They also exhibit a nonzero canonical bracket. Finally (4) implies (3), and a zero bracket has the generator zero, which supplies all stated equivalences. $\square$

For $h = x^n$, $n \geq 2$, write $\beta = [b]$. The pair is $a = [D_{x^{n-2}}]$, $b(x) = x$, $b(Y) = (n-1)Y$, with $[a, \beta] = a \neq 0$ and $a \smile \beta = 0$. For $h = (x^2 + 1)^2$ over $\mathbb{Q}$, take $a = [D_1]$, $b(x) = x^2 + 1$, $b(Y) = 2xY$; the bracket of the two classes is $[D_{2x}] \neq 0$. No splitting field or scalar-extension claim is needed. One should not identify b literally with $\text{ad}_{\pi y}$ in the ambient convention $Y = yh$: that latter derivation has the extra polynomial term $\pi h'' - Lh'$ on Y . Our direct definition avoids that unnecessary identification.

5 Original scope, predecessors, and verification

The recorded Nakayama automorphism is $\nu(x) = x$, $\nu(Y) = Y + h'$ [6, p. 1381]. For nonconstant h , its restriction to $\text{span}_k\{Y, h'\}$ is a nontrivial unipotent Jordan block, so it is not semisimple. For nonzero constant h , it is the identity. Thus Theorem 1 gives both positive and negative answers in the whole original nonsemisimple branch; it is not an extension of a semisimple-Nakayama theorem by omission of a hypothesis.

If $h = 0$ in the original *embedded* definition, the generated algebra is $k[x]$. Here $\text{HH}^0 = k[x]$, $\text{HH}^1 = k[x]\partial_x$, and higher cohomology vanishes. Set $\Delta(f\partial_x) = -f'$, $\Delta(k[x]) = 0$. The Leibniz rule verifies (1) in degrees $(0, 1)$ and $(1, 1)$, and the other cases vanish; $\Delta^2 = 0$. This separately answers that degenerate specialization. Setting $h = 0$ in the *abstract two-generator presentation* gives a polynomial plane, which is a different algebra.

Kowalzig–Krähmer’s ordinary-cohomology theorem [4, Theorem 1.7] assumes semisimplicity of the modular automorphism; its construction uses twisted-coefficient homology in a duality. It does not classify the present nonsemisimple branch. Herscovich–Li’s earlier no-generator theorem [3] concerns the finite-dimensional three-generator Fomin–Kirillov algebra, not this Ore-domain family, and already uses the stronger notion of generator without square-zero. The recent positive theorem for truncated basic cycle algebras [2, Theorem 7.11] concerns another family; its Criterion 7.7 includes commuting odd generators, unlike the obstruction pair here. Neither theorem is contradicted or extended to an unstated class.

Our bounded literature investigation through 22 October 2026 found no authenticated prior statement of this complete A_h classification. The structural classification and the general no-generator phenomenon are inherited, and no exhaustive novelty or current-openness certificate is asserted. We do not treat a dependency DOI as a prior publication of this theorem. The separate general-algebra and Poisson questions in [5] are not claimed answered.

The source supplement includes a portable standard-library exact checker for representative rational-polynomial identities and normalized-bar telescoping, with deliberately false controls. It is newly written supporting diagnostics, not execution of the original claimant’s code and not a proof of the universal theorem. The proof is the argument above, subject to the explicitly named published structural inputs. No formal proof-assistant verification is claimed.

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