Maximal norm discontinuity for a bounded-region continuum fermion interaction

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Research note, version 1.0 — 14 October 2026

Abstract

For continuum fermions in three dimensions, we give one fixed smooth compactly supported pair potential and one fixed bounded interaction region for which the Heisenberg orbit of a fixed annihilation operator is discontinuous in operator norm. Explicit finite-particle Slater states and times tending to zero witness norm differences tending to the maximal value 2. The construction works both with diagonal density pairs and with normal ordering, without ultraviolet smearing. A fixed-cutoff commutator estimate, a Slater second-moment identity, and a two-sector comparison supply a self-contained proof. Continuity on every fixed vector and in every fixed normal state remains valid.

Question and scope. Oberwolfach Report 7/2025, printed pp. 365 and 371 [1], distinguishes pointwise operator-norm continuity from strong-operator continuity and asks for a potential and states violating the former for an unregularized interaction restricted to a bounded region. We answer that existential question in dimension three, with kinetic energy on all of $\mathbb{R}^3$. The Hartree discussion preceding the fermionic question concerns a separate bosonic model. We impose no fixed-density or fixed-energy restriction on the varying witnesses.

Siebert's earlier 2026 preprint [3] proves full-space norm discontinuity by a high-density cluster and localized phase argument. That proof adapts directly to the bounded symmetric cutoff kernel, yielding the qualitative conclusion here. Its Proposition 2.6 also gives the maximal limit 2 for the full-space constant-number reference. Our explicit free-Slater comparison transfers that limit to the fixed spatial cutoff. The audit also records an additional energy-bootstrap derivation using the earlier localized phase mechanism; neither cutoff deduction is a theorem literally stated in the cited preprint. Siebert's 2025 construction [2] uses an extended algebra, with norm continuity after time averaging. This note claims neither first priority nor that the 2025 question remained open when this proof was prepared. The accompanying verification archive records the bounded primary-source priority audit and the precise relation to earlier mechanisms.

Model and statement. Let $\mathfrak{h} = L^2(\mathbb{R}^3)$, $h = -\Delta$, and $\mathcal{F} = \bigoplus_{n\geq 0} \bigwedge^n \mathfrak{h}$. Inner products are linear in the second variable; exterior products use normalized Slater determinants. Write $a(f)$ and $a^*(f)$ for the standard CAR operators, with $\{a(f), a^*(g)\} = \langle f, g\rangle \mathbf{1}$ and $\|a(f)\| = \|f\|$. The ordering flag $\kappa \in \{0, 1\}$ below is distinct from an ultraviolet regularization parameter; neither model is smeared.

Theorem 1. *Set $\Lambda = B(0,4)$ and $L = d\Gamma(\mathbf{1}_\Lambda)$. There exist a fixed real radial $V \in C_c^\infty(\mathbb{R}^3)$, $0 \le V \le 1$, equal to one on $\Lambda - \Lambda$, and a fixed normalized $f \in C_c^\infty(B(0,1/2))$, with the following property. For*

$$H_n^{(\kappa)} = T_n + L_n(L_n - \kappa), \qquad T_n = \sum_{j=1}^n (-\Delta_{x_j}), \quad L_n = \sum_{j=1}^n \mathbf{1}_\Lambda(x_j), \tag{1}$$

and their self-adjoint maximal direct sum $H^{(\kappa)}$, there are explicit unit Slaters, indexed by $N \ge 1$, $\Psi_N \in \bigwedge^N \mathfrak{h}$ and $\Xi_N = a^*(f)\Psi_N \in \bigwedge^{N+1} \mathfrak{h}$ such that

$$t_N^{(\kappa)} = \frac{\pi}{2N + 1 - \kappa}, \qquad D_N^{(\kappa)} = e^{it_N^{(\kappa)} H^{(\kappa)}} a(f) e^{-it_N^{(\kappa)} H^{(\kappa)}} - a(f) \tag{2}$$

satisfy

$$\|D_N^{(\kappa)} \Xi_N + 2\Psi_N\| \longrightarrow 0, \qquad \|D_N^{(\kappa)} \Xi_N\| \longrightarrow 2, \qquad \|D_N^{(\kappa)}\| \longrightarrow 2. \tag{3}$$

The same operator-norm limit holds for the creation difference. The normal vector states $\omega_N(B) = \langle \Xi_N, B\Xi_N \rangle$ satisfy $\omega_N((D_N^{(\kappa)})^* D_N^{(\kappa)}) \to 4$.

Fixed potential and sector domains. Put $b(s) = e^{-1/s}$ for $s > 0$ and $b(s) = 0$ for $s \le 0$, and choose

$$V(x) = \frac{b(81 - |x|^2)}{b(81 - |x|^2) + b(|x|^2 - 64)}. \tag{4}$$

The denominator is positive everywhere. This smooth function equals one on $\overline{B(0,8)}$ and vanishes outside $B(0,9)$. Thus the rigorous sector realization of the density-density interaction in $\Lambda \times \Lambda$ is

$$\sum_{i,j=1}^n \mathbf{1}_\Lambda(x_i)\mathbf{1}_\Lambda(x_j)V(x_i - x_j) = L_n^2.$$

Omitting the diagonal terms gives $L_n(L_n - 1)$, since $V(0) = 1$. We use the coefficient 1 in this ordered-pair sum, without a factor $1/2$. These multiplication operators define the field-expression conventions without multiplying distributions. Each is real, symmetric in the particle coordinates, and bounded by n^2. Hence (1) is self-adjoint on $D(T_n) = H^2(\mathbb{R}^{3n}) \cap \bigwedge^n \mathfrak{h}$. The vacuum sector is zero. The full domain is

$$D(H^{(\kappa)}) = \left\{ (\eta_n) \in \mathcal{F} : \eta_n \in D(T_n),\ \sum_n \|H_n^{(\kappa)} \eta_n\|^2 < \infty \right\}.$$

No uniform Fock-space bound on the interaction is presumed. Nor is invariance of the CAR algebra under the interacting evolution needed for this norm comparison in $B(\mathcal{F})$.

Packed Slaters. Choose $f(x) = c_f b(1 - 9|x|^2)$, with $c_f > 0$ normalizing its norm. Its support is $\overline{B(0,1/3)}$. Let $Q = 2e_1 + (-1/4, 1/4)^3 \subset B(0,3)$, disjoint from $\operatorname{supp} f$, and normalize $\varphi(x) = c_\varphi \prod_{k=1}^3 b(1 - 16(x_k - 1/2)^2)$. This belongs to $C_c^\infty((0,1)^3)$. For $m_N = \lceil N^{1/3} \rceil$, subdivide Q into m_N^3 cubes of side $\ell_N = (2m_N)^{-1}$. Choose the first N lower corners $c_{j,N}$ in lexicographic order, and set

$$\psi_{j,N}(x) = \ell_N^{-3/2} \varphi((x - c_{j,N})/\ell_N), \quad \Psi_N = \psi_{1,N} \wedge \cdots \wedge \psi_{N,N}, \quad \Xi_N = a^*(f)\Psi_N.$$

Disjoint supports give orthonormal orbitals and $a(f)\Psi_N = 0$, so both Slaters have norm one. Since $m_N \le 2N^{1/3}$,

$$\sum_{j=1}^{N} \|\nabla\psi_{j,N}\|^2 = N\ell_N^{-2}\|\nabla\varphi\|^2 \le 16\|\nabla\varphi\|^2 N^{5/3}.$$

For either Slater with its actual particle number $n = N$ or $N+1$, and its initial orbitals $g_1, \dots, g_n$, therefore

$$\sum_{j=1}^{n} \|g_j\|_{H^1}^2 \le C_0 n^{5/3}, \qquad C_0 = 1 + 16\|\nabla\varphi\|^2 + \|\nabla f\|^2. \tag{5}$$

All initial supports lie in $B(0,3)$.

Uniform free leakage. Use the single fixed cutoff

$$\chi(x) = \frac{b(49/4 - |x|^2)}{b(49/4 - |x|^2) + b(|x|^2 - 9)}.$$

It lies in $C_c^\infty(\Lambda)$ and equals one on $B(0,3)$. With $q = \mathbf{1}_{\Lambda^c}$,

$$\begin{aligned} [h,\chi]u &= -(\Delta\chi)u - 2\nabla\chi\cdot\nabla u, \\ \|[h,\chi]u\| &\le C_\chi\|u\|_{H^1}, \qquad C_\chi = (\|\Delta\chi\|_\infty^2 + 4\|\nabla\chi\|_\infty^2)^{1/2}. \end{aligned} \tag{6}$$

For any initial orbital g, differentiate $e^{-i(s-r)h}\chi e^{-irh}g$ on its H^2 domain to obtain

$$(e^{-ish}\chi - \chi e^{-ish})g = -i\int_0^s e^{-i(s-r)h}[h,\chi]e^{-irh}g\,dr. \tag{7}$$

The free group preserves H^2 and is isometric on H^1; multiplication by χ preserves H^2. Thus the identity is justified for every packed smooth orbital, while its bound uses only the H^1 norm. From $\chi g = g$ and $q\chi = 0$, for every real s,

$$\|qe^{-ish}g\| \le C_\chi|s|\|g\|_{H^1}. \tag{8}$$

The indicator q is never differentiated.

Second moment of the leaked number. Set $M_n = n - L_n = d\Gamma(q)$ on the n-particle sector. For either initial Slater Φ_n, free evolution $\Phi_n(s) = e^{-isT_n}\Phi_n$ remains a normalized Slater. Let P be its rank-n occupied projection and $\lambda = \mathrm{Tr}(Pq)$. Expanding the determinant's two-particle marginal gives

$$\langle M_n\rangle = \lambda, \qquad \langle M_n^2\rangle = \lambda^2 + \lambda - \mathrm{Tr}(PqPq). \tag{9}$$

Indeed the diagonal terms are $\sum_j\langle g_j, qg_j\rangle$, as $q^2 = q$; the terms with $j \ne k$ are $\langle g_j, qg_j\rangle\langle g_k, qg_k\rangle - |\langle g_j, qg_k\rangle|^2$. Their sum proves (9). All traces are finite, and $\mathrm{Tr}(PqPq) \ge 0$. With $\beta = C_\chi\sqrt{C_0}$, (5) and (8) yield

$$\begin{aligned} \lambda_n(s) &\le \beta^2 s^2 n^{5/3}, \\ \|M_n\Phi_n(s)\| &\le \sqrt{\lambda_n(s) + \lambda_n(s)^2} \le \beta|s|n^{5/6} + \beta^2 s^2 n^{5/3}. \end{aligned} \tag{10}$$

This variance estimate concerns Slaters; it is not asserted for general correlated or interacting states.

Two-sector comparison. Introduce only as a reference $K_n^{(\kappa)} = T_n + n(n-\kappa)$. Since M_n has spectrum $\{0, \dots, n\}$,

$$H_n^{(\kappa)} - K_n^{(\kappa)} = -M_n(2n - \kappa - M_n), \qquad \|(H_n^{(\kappa)} - K_n^{(\kappa)})\eta\| \le 2n\|M_n\eta\|. \tag{11}$$

The difference is bounded on each sector, so strong Duhamel holds there. Reference evolution differs from free evolution by a scalar phase. Applying (10), for either Φ_n and every real t, gives

$$\begin{aligned} E_n(t) &:= \|(e^{-itH_n^{(\kappa)}} - e^{-itK_n^{(\kappa)}})\Phi_n\| \le 2n \int_0^{|t|} \|M_n\Phi_n(\mathrm{sgn}(t)s)\|\, ds \\ &\le \beta n^{11/6}|t|^2 + \tfrac{2}{3}\beta^2 n^{8/3}|t|^3. \end{aligned} \tag{12}$$

Thus for *both* $n = N$ and $n = N+1$, since $n \le 2N$ and $t_N^{(\kappa)} \le \pi/(2N)$,

$$E_n(t_N^{(\kappa)}) = O(N^{-1/6} + N^{-1/3}) \longrightarrow 0. \tag{13}$$

The constants are independent of N and κ. We have used the Slater estimate only on free reference states, with no assumption that interacting evolution preserves Slaters or that a changing many-particle free state stays close to its initial vector.

Phase and norm witness. Let $A = a^*(f)$ and, suppressing the fixed ordering flag, write $F_N^H(t) = \langle \Xi_N, e^{itH} A e^{-itH} \Psi_N \rangle$. Replacing the separately evolved bra and ket by their reference evolutions, unitarity and $\|A\| = 1$ imply

$$|F_N^H(t) - F_N^K(t)| \le E_{N+1}(t) + E_N(t). \tag{14}$$

For $K = d\Gamma(h) + \mathcal{N}(\mathcal{N} - \kappa)$, free covariance and the CAR give exactly

$$\begin{aligned} F_N^K(t) &= e^{it(2N+1-\kappa)} \langle a^*(f)\Psi_N, a^*(e^{ith}f)\Psi_N \rangle \\ &= e^{it(2N+1-\kappa)} \langle f, e^{ith} f \rangle. \end{aligned} \tag{15}$$

The last equality uses $a(f)\Psi_N = 0$; it does not require $e^{ith}f$ to remain orthogonal to the occupied orbitals. At $t = t_N$ the scalar phase is -1, and $\langle f, e^{it_N h} f \rangle \to 1$. Consequently $F_N^H(t_N) \to -1$, whereas $F_N^H(0) = 1$. Taking adjoints,

$$\langle \Psi_N, D_N \Xi_N \rangle = \overline{F_N^H(t_N) - 1} \longrightarrow -2.$$

The universal upper bound $\|D_N\| \le 2$ now proves both norm limits. More quantitatively, since $f \in D(h)$,

$$2 \ge \|D_N \Xi_N\| \ge 2 - E_N(t_N) - E_{N+1}(t_N) - t_N \|hf\|.$$

Finally $\|D_N\Xi_N + 2\Psi_N\|^2 \le 8 + 4\Re\langle \Psi_N, D_N\Xi_N \rangle \to 0$. This proves the first limit in (3) and the vector-state conclusion, and completes the proof of Theorem 1.

Topology and boundaries. For every fixed vector η, any self-adjoint H and bounded B obey

$$\|(e^{itH} B e^{-itH} - B)\eta\| \le \|B\|\|(e^{-itH} - 1)\eta\| + \|(e^{itH} - 1)B\eta\| \to 0.$$

For $B = a(f)$, this also implies $\mathrm{Tr}(\rho D_t^* D_t) \to 0$ for every fixed positive trace-class state ρ of trace one, by its spectral decomposition and the uniform bound 4. The witnesses in the theorem vary with N. The result concerns the chosen ball and potential; it does not assert all potentials or regions, fixed-density thermodynamic behavior, or CAR non-invariance. The packing exponents $-1/2 + 1/d$ and $-1 + 2/d$ do not settle $d = 1, 2$.

Verification and disclosure. The accompanying source archive contains the unchanged original mathematical packet, independent analytic and source audits, finite algebra/numerical controls, and publication verification safeguards. Finite controls supplement the written universal proof; numerical diagnostics are not interval-certified proofs. Source hashes are consistency fingerprints, not signatures. AI tools were used extensively in developing, checking, and preparing this work, including adversarial reviews by separate AI agents. This preprint is unrefereed, and no independent human peer review is claimed.

References

[1] Oberwolfach Report 7/2025, *Mini-Workshop: New Directions in Correlated Quantum Systems*, doi:10.4171/owr/2025/7, printed pp. 364–365 and 371.

[2] O. Siebert, *On the thermodynamic limit of interacting fermions in the continuum*, Letters in Mathematical Physics **115**, 33 (2025), doi:10.1007/s11005-025-01919-y; arXiv:2409.10495v2, 24 February 2025.

[3] O. Siebert, *Discontinuity of Continuum Fermion Dynamics*, arXiv:2609.34402v1, submitted 28 September 2026.