

Ambient Dependence of Extremal-Exchangeable Independence

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Abstract

Liebscher asked whether independence defined by an extreme exchangeable free-product state needs to retain its ambient algebra. We give a state-preserving inclusion of a four-point algebra into $M_4(\mathbb{C})$ for which two distinct proper coordinate subalgebras, intersecting only in the scalars, are dependent downstairs and independent upstairs. Positivity forces every downstairs witness to be the same nontrivial character mixture; upstairs a pure vector state supplies a pure folding witness. Pure folding is an established construction, credited below. We also exhibit an extreme exchangeable algebraic state with no positive extension to a larger algebra, addressing the source's separate non-lifting concern. In contrast, extreme invariant states do lift along equivariant unital C^* -inclusions. The definition is unrestricted: extremality is among all exchangeable states, without a fixed-marginal, faithfulness or trace condition.

1 The original question and convention

Let A be a unital complex $*$ -algebra and let φ be a positive unital linear functional. Write $\mathcal{F}(A) = *_{i \in \mathbb{N}} A$ for the unital algebraic free product, with injections j_i , and let $\mathcal{E}(A)$ be its convex set of states invariant under finite permutations of the copies. In the C^* -category we use the universal unital C^* -free product and continuous states.

For unital $*$ -subalgebras $B, C \subseteq A$, Liebscher's definition on printed page 540 of [1] declares them independent when some $\omega \in \text{ext } \mathcal{E}(A)$ satisfies

$$\omega \circ ((j_1|_B) * (j_2|_C)) = \varphi \circ (\iota_B * \iota_C) \quad \text{as states on } B * C. \quad (1)$$

Here ι_B, ι_C are the original inclusions, and the maps from $B * C$ are supplied by the universal property. Equality means the entire joint law. Extremality is in $\mathcal{E}(A)$, not in a slice with prescribed marginals. Item (1) asks whether retaining the ambient algebra is necessary, and motivates this by a possible failure of extreme-state lifting. We distinguish those two issues.

2 A finite-dimensional counterexample

Theorem 1. Put $A = \mathbb{C}^{\{00,01,10,11\}}$, let $p(s, t) = s$, $q(s, t) = t$, and set

$$B = \text{span}\{1, p\}, \quad C = \text{span}\{1, q\}, \quad \varphi = \frac{1}{2}\delta_{00} + \frac{1}{2}\delta_{11}.$$

Embed A diagonally in $M_4(\mathbb{C})$, in the displayed point order, and let

$$u = (e_{00} + e_{11})/\sqrt{2}, \quad \varphi'(T) = u^* T u.$$

This inclusion preserves the state. The subalgebras B, C are distinct, proper and have scalar intersection. They are not independent in (A, φ) , but are independent in $(M_4(\mathbb{C}), \varphi')$ under (1), in both free-product categories.

Proof. The functions $1, p, q, pq$ form a basis of A . Evaluating $a + bp = c + dq$ at $00, 10, 01$ gives $a = c$, $b = d = 0$. The stated subalgebra properties follow. Restriction of the vector state to the diagonal gives exactly φ .

Suppose that an exchangeable state ω satisfies (1) downstairs. In its GNS representation put $P_i = \pi(j_i(p))$, $Q_i = \pi(j_i(q))$, with cyclic vector Ω . The joint law and exchangeability imply, for every $i \neq j$,

$$\langle \Omega, P_i \Omega \rangle = \langle \Omega, Q_j \Omega \rangle = \langle \Omega, P_i Q_j \Omega \rangle = \langle \Omega, Q_j P_i \Omega \rangle = \frac{1}{2}.$$

Consequently

$$\|(P_i - Q_j)\Omega\|^2 = 0 \quad (i \neq j).$$

Choosing a third index shows that all $P_i \Omega$ and $Q_i \Omega$ are the same vector v . Each projection R in this family satisfies $R\Omega = v$ and $Rv = R^2\Omega = v$. Thus every nonempty finite word in the generators sends Ω to v and has moment $1/2$. These words and the unit span the algebraic free product.

Let χ_0, χ_1 evaluate every copy at 00, 11, respectively. The preceding calculation proves, for every possible witness,

$$\omega = \frac{1}{2}\chi_0 + \frac{1}{2}\chi_1.$$

The two characters are distinct exchangeable states. Therefore ω is not extreme. Equality on dense algebraic words gives the same conclusion for the universal C^* -free product. This excludes every downstairs witness, rather than just a chosen one.

Upstairs, let $f : \mathcal{F}(M_4) \rightarrow M_4$ fold all copies identically. The vector state φ' is pure, and its pullback $\omega' = \varphi' \circ f$ is pure by Lemma 2. The folding map is unchanged by permutations, so ω' is exchangeable and hence extreme in the exchangeable state space. For every word from $B * C$, folding multiplies the original diagonal matrices in their original order. Restriction of φ' to A proves (1) on the entire algebra. Moreover $\omega' \circ j_i = \varphi'$ on every full copy. $\square$

The algebraic GNS calculation above needs only Cauchy–Schwarz. For clarity, its null space is a left ideal: if $\eta(x^*x) = 0$, Cauchy–Schwarz applied to x and a^*ax gives $\eta(x^*a^*ax) = 0$, so ax is null. Projection multiplication therefore acts on the quotient domain. No null-vector identity was silently promoted to an operator identity.

Lemma 2. *A pure state pulls back to a pure state along a surjective unital $*$ -homomorphism. A unit vector state on $M_d(\mathbb{C})$ is pure. Both statements hold for positive algebraic states.*

Proof. In a convex decomposition of $\eta \circ f$, positivity forces each constituent to vanish on x^*x for $x \in \ker f$. Cauchy–Schwarz then makes it vanish on x . Each constituent descends to a unital positive functional on the quotient: lift any a to z , and evaluate a^*a as z^*z . Purity of η makes the decomposition trivial. For a vector state, let $e = uu^*$. The zero expectation of $1 - e$ in every constituent kills all blocks outside e , by Cauchy–Schwarz. Since $eTe = (u^*Tu)e$ and each constituent has value one on e , each is the original vector state. $\square$

Pure folding is the known maximal-amalgamation construction: Dykema, Köstler and Williams [2, Proposition 8.7] characterize its quantum symmetric extremality, and their Theorem 8.1 relates this to ordinary symmetric extremality. Lemma 2 gives the needed special case directly, with the stronger conclusion of purity among all states.

The half weight is inessential. Replacing it by $(1 - t)\delta_{00} + t\delta_{11}$, $0 < t < 1$, and using $\sqrt{1 - t}e_{00} + \sqrt{t}e_{11}$, gives the same proof with moment t . At $t = 0, 1$ the downstairs mixture becomes a character. If coincident subalgebras are allowed, the smaller example $A = B = C = \mathbb{C}^2 \subset M_2$ works with the faithful downstairs fair state and its coherent pure extension.

3 The literal lifting concern

Theorem 3. *There is a unital inclusion of algebraic $*$ -algebras for which an extreme exchangeable state on the smaller free product has no positive extension to the larger one.*

Proof. Take $R = \mathbb{C}[x] \subset S = \mathbb{C}[x, x^{-1}]$, with both x and x^{-1} self-adjoint. Evaluation at zero in every copy gives a character χ on $\mathcal{F}(R)$, hence a pure exchangeable state. A positive unital extension η to $\mathcal{F}(S)$ would satisfy, in the first copy, $X^* = X$, $Y^* = Y$, $XY = 1$, $\eta(X^2) = 0$. Cauchy–Schwarz gives the contradiction

$$1 = |\eta(XY)|^2 \leq \eta(X^2)\eta(Y^2) = 0.$$

$\square$

Both R, S admit the common state given by evaluation at one. The forbidden χ is a different state; the assertion concerns the unrestricted extreme-state sets in the source, not failure to extend that specified common marginal.

Proposition 4. *For an equivariant unital C^* -inclusion $D \subset E$ under the group of finitely supported permutations of $\mathbb{N}$, every extreme invariant state of D has an extreme invariant extension to E .*

Proof. Extend the state to E by the C^* -state extension theorem. Average the extension over S_n and take a weak-* cluster point, using a subnet when necessary. Every fixed finite permutation fixes all sufficiently late averages; the limit is an invariant extension. Its compact convex extension fiber is a face of the invariant state space of E , since restriction of a decomposition would decompose the extreme state of D . An extreme point of this nonempty compact face is extreme in the whole space. $\square$

The canonical map between universal unital free products of a C^* -inclusion is injective. One may check this by extending each copy representation using state extension and cyclic GNS dilations, padding them to a common Hilbert space, and applying the universal property. The verification package gives this argument in detail. Thus Proposition 4 applies to those products. Extreme states can lift while their restrictions fail to remain extreme. Theorem 1 proves ambient dependence in that opposite direction; Theorem 3 concerns the algebraic category, where C^* -extension and compactness are unavailable.

4 Scope, attribution and verification

The printed free-product definition is the target. We impose no global commutativity on its state algebra merely because A is commutative. A classical commutative/tensor interpretation requires a different coproduct or states factoring through a commutative quotient; it is not assumed here.

The counterexample uses nonfaithful states, as permitted by the definition, and no tracial witness. It does not address faithful-only, tracial-only, trivial-tail or separately specified normal W^* -definitions. Adding a prescribed full marginal while retaining extremality in $\mathcal{E}(A)$ leaves the upstairs witness valid. Changing the convex set to the fixed-marginal slice changes the result: the forced downstairs mixture is extreme in that slice, as independently proved in the supporting audit.

The known pure-folding mechanism is credited above. Our source audit does not certify historical-first priority, and we make no such claim. The note supplies explicit counterexamples and full deductions for the stated original question, rather than resolving all the other questions in the workshop contribution.

The source archive contains the manuscript, immutable authored input, independent proofs and rational verification scripts, source hashes and the priority audit. Finite checks supplement the all-state, all-word and infinite-copy proofs; they are not formal certification. AI tools were used extensively in proof development, reproduction, adversarial review and preparation. All listed reviews were AI-assisted. This preprint is unrefereed and has not received human peer review.

References

- [1] V. Liebscher, On Quantum Independence, pp. 540–541 in B. V. R. Bhat, U. Franz and M. Skeide (organizers), *Mini-Workshop: Product Systems and Independence in Quantum Dynamics*, Oberwolfach Reports **6** (2009), 493–548. [doi:10.4171/OWR/2009/09](https://doi.org/10.4171/OWR/2009/09).
- [2] K. J. Dykema, C. Köstler and J. D. Williams, Quantum symmetric states on free product C^* -algebras, *Trans. Amer. Math. Soc.* **369** (2017), 645–679. [doi:10.1090/tran6661](https://doi.org/10.1090/tran6661); [arXiv:1305.7293v3](https://arxiv.org/abs/1305.7293v3).

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