

A two-cube cubulation of the AIM free-by-cyclic example

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Abstract

We construct a finite locally CAT(0) unit cube complex with one vertex, seven edges, eight squares, and two three-dimensional cubes, whose fundamental group is the mapping torus of $a \mapsto a, b \mapsto ab, c \mapsto bcb$. Its universal cover gives a proper cocompact cubical action of the exact group in AIM Question 5.2. All attaching maps, signed vertex links, and reversible presentation changes are given explicitly. We also explain a qualitative existence deduction from earlier WPD, Morse, and cubical completion theorems. The contribution presented here is the explicit small finite model; no historical firstness or minimality claim is made.

1 The exact statement

Let $F = F(a, b, c)$ and let $\phi \in \text{Aut}(F)$ be given by

$$\phi(a) = a, \quad \phi(b) = ab, \quad \phi(c) = bcb.$$

Indeed its inverse sends a to a , b to $a^{-1}b$, and c to $b^{-1}acb^{-1}a$; composition in either order freely reduces to the respective basis generators. We use the convention

$$G = F \rtimes_{\phi} \mathbb{Z} = \langle a, b, c, t \mid tat^{-1} = a, tbt^{-1} = ab, tct^{-1} = bcb \rangle. \quad (1)$$

The final b is positive. AIM Question 5.2, attributed to Rylee Lyman, asks whether this group acts properly and cocompactly on a CAT(0) cube complex [1]. The workshop report discusses the different formula $b \mapsto aba, c \mapsto bcb$ and an earlier three-dimensional candidate [2, p. 2]. We keep that related example distinct.

Theorem 1. *There is a finite locally CAT(0) unit cube complex Z of dimension three with cell counts $(1, 7, 8, 2)$ and $\pi_1 Z \cong G$. Consequently G acts freely, properly, and cocompactly by cubical isometries on the CAT(0) cube complex $\tilde{Z}$.*

Here a cube complex allows isometric identifications of boundary faces; characteristic cubes in the quotient need not be embedded. This convention is essential for the stated cell counts. The flag-link criterion for this convention is Leary's Theorem C.9 [6, p. 33]. His Theorem C.4 (p. 31) ensures that a CAT(0) universal cover is cubical under the stricter embedded-cube convention; Corollary C.11 (p. 33) gives a strict cubical quotient after the second cubical subdivision, if desired. The counts in Theorem 1 are before subdivision.

The construction uses a finite cyclic local isometry into a two-square base. We verify the quotient links directly and identify the entire fundamental group. No minimum dimension, minimum cell

count, or virtual specialness conclusion is asserted. Earlier tools also imply existence through the target-specific deduction in Section 5. A bounded source audit on October 12, 2026 found no prior documented exact answer or directly covering geometric-cubulation theorem in the sources examined. This is not a certificate of historical firstness or absolute novelty.

2 The two-square base and cyclic domain

In attaching words, capitals denote inverse oriented edges. Vertex names A, B, C, D are independent of this convention. Let K have one vertex, three oriented unit edges x, y, b , and two unit squares with boundary words

$$xyXY, \quad bxBY. \quad (2)$$

Thus $H = \pi_1 K = \langle x, y, b \mid [x, y] = 1, bxb^{-1} = y \rangle$. For an edge h , write h^+ and h^- for the outgoing and incoming germs. The link L_K has six vertices and precisely the eight edges

$$\{x^\epsilon, y^\delta\} \quad (\epsilon, \delta \in \{+, -\}), \quad \{b^-, x^+\}, \{b^-, x^-\}, \{b^+, y^+\}, \{b^+, y^-\}. \quad (3)$$

It is a simple bipartite graph, with parts $\{x^+, x^-, b^+\}$ and $\{y^+, y^-, b^-\}$. It has no triangles and is therefore flag, so K is locally CAT(0).

The assignments

$$\lambda(x) = y, \quad \lambda(y) = x, \quad \lambda(b) = B \quad (4)$$

extend to a cubical isometric involution of K . They take the two boundary words in (2) to $yxYX$ and $BybX$, respectively. The first is a dihedral reparametrization of the first square; the second is a cyclic rotation of the inverse of the second boundary. The corresponding square isometries agree with (4) on their faces. Their squares are the identity, so they define the claimed involution, including on cells.

Define Q by four distinct vertices A, B, C, D and the following six unit edges. The final column specifies a cubical map $j : Q \rightarrow K$.

edge	initial vertex	terminal vertex	j -label
p	B	A	x
q	A	C	y
r	D	C	x
s	B	D	y
d	A	B	b
e	C	A	b

Attach two squares with words

$$pqr^{-1}s^{-1} \quad \text{based at } B, \quad dpe^{-1}q^{-1} \quad \text{based at } A. \quad (5)$$

Their characteristic vertex cycles are (B, A, C, D) and (A, B, A, C) . In particular, the two occurrences of A in the second square must both be retained in corner calculations. The labels take these boundaries to $xyXY$ and $bxBY$, so j is defined on squares by unit-square isometries.

The complete signed links are given below. A path lists all link vertices and all edges; there are no additional edges or higher simplices. The last two columns record their images in L_K under the two endpoint maps.

vertex	path in Lk_Q	image under j	image under λj
A	$e^- - p^- - q^+ - d^+$	$b^- - x^- - y^+ - b^+$	$b^+ - y^- - x^+ - b^-$
B	$d^- - p^+ - s^+$	$b^- - x^+ - y^+$	$b^+ - y^+ - x^+$
C	$r^- - q^- - e^+$	$x^- - y^- - b^+$	$y^- - x^- - b^-$
D	$r^+ - s^-$	$x^+ - y^-$	$y^+ - x^-$

For example, the two A -corners of the second square supply the distinct edges $\{d^+, q^+\}$ and $\{p^-, e^-\}$, while the first square supplies $\{p^-, q^+\}$. Every row maps injectively to L_K . Comparison with (3) shows that each image is a full induced subcomplex: every base-link edge between listed vertices is listed. Consequently Q is locally $CAT(0)$, and j and λj are cubical local isometries, by the injective-full-link characterization.

The edges p, q, s form a spanning tree. On collapsing them, the first square kills r and the second gives $d = e$. Hence

$$\pi_1(Q, A) = \langle dp \rangle \cong \mathbb{Z}, \quad j_*(dp) = bx, \quad (\lambda j)_*(dp) = By. \quad (6)$$

The maps are injective on fundamental groups; alternatively bx has infinite order because the homomorphism $H \rightarrow \mathbb{Z}$ sending b to 1 and x, y to 0 sends bx to 1.

3 The finite quotient and its flag link

Form the double mapping cylinder

$$Z = K \cup_{j, \lambda j} (Q \times [0, 1]), \quad (7)$$

where $(q, 0)$ is attached by j and $(q, 1)$ by λj . Give the interval length one and the product its cubical structure. Face identifications come from the genuine product and the two cubical maps. They need not be globally injective. Denote the vertical edges by z_A, z_B, z_C, z_D , directed from the j end to the λj end. These are four distinct edges, although all their endpoints become the sole vertex of K . The second product cube has z_A at two distinct boundary positions, just as the second square of Q has two A -corners.

There are three base edges and four vertical edges, two base squares and six edge-product squares, and two square-product three-cubes. Thus

$$(\#Z^{(0)}, \#\text{edges}, \#\text{squares}, \#\text{three-cubes}) = (1, 7, 8, 2). \quad (8)$$

The Euler characteristic is $1 - 7 + 8 - 2 = 0$.

Let L be the unique vertex link of Z . It consists of L_K and eight cones with apices z_v^+ and z_v^- , $v \in \{A, B, C, D\}$. Their bases are the j and λj link images in the table, respectively. This description follows corner by corner from the product: an edge-product square adds an apex-base edge, and a square-product cube adds the cone triangle over each corresponding square corner. No two apices are adjacent, since a product-cube corner has only one interval direction. The eight apices are distinct germs of the four vertical edges.

The link maps in the table are injective, so no cone has repeated vertices, edges, or triangles. Distinct cones have distinct apices. At a repeated A -corner the two triangles from the second product cube have distinct base edges, as displayed above; they are not a repeated simplex. Therefore L is an actual simplicial complex, not merely a graph with prescribed clique counts.

If a clique of L contains no apex, it lies in the flag complex L_K . Otherwise it contains exactly one apex. Its other vertices lie in the full cone base, and their base-link clique spans a simplex in

that base. Adjoining the apex gives a simplex in the cone. Thus L is flag. Its counts provide an additional check: it has $6 + 8 = 14$ vertices, $8 + 2(4 + 3 + 3 + 2) = 32$ edges, and $2(3 + 2 + 2 + 1) = 16$ triangles, with no four-clique. Leary's general cube-complex criterion [6, Theorem C.9, p. 33] now shows that Z is locally CAT(0) and that its universal cover is CAT(0).

4 The fundamental group and geometric action

We record a direct presentation computation to verify the exact group, including the sign of the last b . Besides (2), the six edge-product squares have relators

$$\begin{aligned} xz_A Y z_B^{-1}, \quad yz_C X z_A^{-1}, \quad xz_C Y z_D^{-1}, \\ yz_D X z_B^{-1}, \quad bz_B b z_A^{-1}, \quad bz_A b z_C^{-1}. \end{aligned} \tag{9}$$

Attaching three-cells does not change the fundamental group. Write $z = z_A$. The p, q, s relations eliminate the other vertical generators:

$$z_B = xzY, \quad z_C = Yzx, \quad z_D = YxzYx. \tag{10}$$

The r relation is then redundant using $xy = yx$. The d relation is $z^{-1}(bx)z = By$; the e relation is $z^{-1}(yb)z = xB$. These agree, because $bx B = y$ gives $yb = bx$ and $xB = By$. Each elimination is a reversible Tietze transformation, so

$$\pi_1 Z = \langle x, y, b, z \mid xy = yx, \, bx B = y, \, z^{-1}(bx)z = By \rangle. \tag{11}$$

Make the invertible change of generators

$$t = x, \quad a = xy^{-1}, \quad c = (zx)^{-1}; \quad x = t, \quad y = a^{-1}t, \quad z = c^{-1}t^{-1}, \tag{12}$$

leaving b fixed. The first two relations of (11) become $[a, t] = 1$ and $btb^{-1} = a^{-1}t$. The latter is equivalent to $tbt^{-1} = ab$. For the third relation put $u = Bx$, $v = bx$. Since $By = xB = xuX$, its successive equivalent forms are

$$\begin{aligned} z^{-1}vz = xux^{-1} &\iff (zx)u(zx)^{-1} = v \\ &\iff c^{-1}(b^{-1}t)c = bt \\ &\iff tc = bcbt \\ &\iff tct^{-1} = bcb. \end{aligned}$$

Thus (11) becomes exactly (1). Explicit inverse substitutions and reversible relations establish an isomorphism, rather than only a homomorphism obtained by checking relators in G .

The cover $\tilde{Z}$ is three-dimensional with finite links and unit Euclidean cube shapes. It is locally finite; its natural complete length metric is proper (equivalently, local compactness and completeness give properness for this geodesic metric). The deck group acts freely by cubical isometries. Covering neighborhoods show that this action is properly discontinuous, and, on the proper cover, that it is a proper action: only finitely many translates of a compact set meet that set. Its quotient is the finite complex Z , so the action is cocompact. The isomorphism above identifies this deck group with the full group G . This proves Theorem 1.

5 An existence deduction from earlier tools

We give a second route to qualitative existence to clarify the relation to prior methods. This route obtains an unspecified finite cyclic local isometry by existing completion theorems; it does not supply the cell counts (8).

Write $A_0 = \langle x, y \mid [x, y] = 1 \rangle \cong \mathbb{Z}^2$ and $H = \langle A_0, b \mid bxb^{-1} = y \rangle$. The HNN normal-form theorem embeds A_0 . In its Bass–Serre tree, vertices are left cosets hA_0 , and an oriented edge e goes from A_0 to bA_0 , with stabilizer $\langle y \rangle$. Set $g = bx$. The homomorphism $\epsilon_b : H \rightarrow \mathbb{Z}$, with $\epsilon_b(b) = 1$ and $\epsilon_b(x) = \epsilon_b(y) = 0$, gives a height on vertices. Since $gA_0 = bA_0$, the edges g^ne concatenate, increasing height at every step. They form a nonbacktracking bi-infinite axis of g . At A_0 , the outgoing edge has stabilizer $\langle y \rangle$ and the incoming edge has stabilizer

$$g^{-1}\langle y \rangle g = x^{-1}b^{-1}\langle y \rangle bx = \langle x \rangle.$$

Their intersection is trivial in A_0 . Hence the pointwise stabilizer of the axis segment from $g^{-1}A_0$ to gA_0 is trivial. Minasyan–Osin [7, Corollary 4.3], in the corrected v6, gives WPD for this specific loxodromic g . No finite-extension claim is used.

Dahmani–Guirardel–Osin [4, Theorem 6.8 and Corollary 2.9] provide a virtually cyclic hyperbolically embedded subgroup $E(g)$ containing g . As $\epsilon_b(g) = 1$, g has infinite order. Sisto’s Theorem 1 [8] then makes g Morse; in particular $C = \langle g \rangle$ is quasiconvex in the word metric on H . The metric required for completion matches exactly: K has one vertex, so $\tilde{K}^{(1)}$ is the Cayley graph for $\{x, y, b\}$, with its edge metric. The two-edge circle $\Gamma \rightarrow K$ reading bx realizes the inclusion $C \hookrightarrow H$, and its lifted image contains C and lies within distance one of it. Geodesics between its vertices also stay uniformly close, by adding endpoint paths of length at most one and using the Morse property. Thus no conversion from CAT(0)-metric quasiconvexity is being assumed.

Ben-Zvi–Kropholler–Lyman [3, Theorems 1.1 and 5.1] now give a finite pointed locally CAT(0) cube complex Q_0 and a cubical local isometry $j_0 : Q_0 \rightarrow K$, through which $\Gamma \rightarrow K$ factors, whose fundamental-group image is exactly C . Their maximal-tree generating set is $\{x, y, b\}$ here. Local-isometry injectivity gives $\pi_1 Q_0 \cong C$, retaining the generator bx , rather than a power of it. The original bx circle itself is not a local isometry: at its intermediate vertex the germs b^-, x^+ span a base-link edge. Completion is what repairs that defect.

Postcomposition with the explicit involution (4) gives the second local isometry λj_0 on the *same* cubical domain, taking the generator to By . Hence $M = K \cup_{j_0, \lambda j_0} (Q_0 \times [0, 1])$ is finite. Its links have the same general structure as those of Z : each interval germ cones off an injective full link image, and distinct interval germs are not adjacent. The clique argument in Section 3 proves that these links are flag. This also gives a direct verification of the local-isometry gluing discussed in Hagen–Wise [5, Remark 3.7]. Because a local isometry into K has domain dimension at most two, $\dim M \leq 3$. Van Kampen gives (11), and (12) identifies $\pi_1 M$ with G . Its finite locally CAT(0) quotient therefore gives another proper cocompact action of the exact group.

The abstract tools in this deduction are prior work. Their availability does not date the discovery or publication of this exact application. The present note makes the element, metric, compatible maps, and group identification explicit. Hagen–Wise’s current general fast-automorphism result [5, Corollary 7.2] supplies free cubical actions; properness and cocompactness require the additional argument above or the explicit model of Sections 2–4.

Verification and disclosure

The accompanying verification package contains three Python standard-library programs for finite combinatorial and algebraic checks. They check the specified face maps, signed links and fullness,

the involution, the fourteen-vertex link, cell counts, and generator substitutions, with validation active in ordinary and optimized Python execution. These finite checks supplement the written proof; they do not constitute a machine proof of the global $CAT(0)$ or deck-action conclusions.

Extensive AI assistance was used in discovery, drafting, and internal independent adversarial checks. This note is unrefereed; there has been no external human peer review and no proof-assistant certification. The author is responsible for the stated claims and their scope.

References

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