An elementary counterexample to a Nullstellensatz for finite-type link invariants

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Abstract

Let V be the ordinary, uncompleted complex algebra of all finite-order finite-type invariants of links. We give a principal ideal with a nonempty actual-link zero set whose vanishing ideal strictly contains its radical. The generator is $g(4g-1)$, where g is the sum of squared pairwise linking numbers. The obstruction is a character on the entire algebra, obtained by evaluating the standard twist-restriction polynomial at a nonintegral parameter. This gives a negative answer to the arbitrary-ideal formulation of Bar-Natan's Problem 2.27, as recorded by Ohtsuki; the separate linking-number-generated ideal question is not decided.

1 The question and the algebra

Bar-Natan asks whether an ideal in the algebra of finite-type link invariants has the same radical as the ideal of invariants vanishing on its common zeros among actual links [1, Problem 2.27, p. 422]. The present note records an elementary counterexample to that literal arbitrary-ideal assertion.

Fix $m \geq 2$, and let X_m consist of oriented, unordered m-component links in S^3, including split links. Let V_m be the unital complex algebra of their ordinary finite-type invariants of some finite order, with pointwise operations. An invariant has order at most d when its positive-minus-negative crossing extension vanishes on every link with $d+1$ double points. Each element has a finite order; there is no common cutoff on all elements and no infinite-order completion. Products of invariants of orders d, e have order at most $d+e$: on a crossing-resolution cube their multilinear representatives have degrees at most d, e, and the product has degree at most $d+e$.

For an ideal $A \subset V_m$, set

$$Z_{\mathrm{act}}(A) = \{L \in X_m : a(L) = 0 \text{ for all } a \in A\}, \qquad J_{\mathrm{act}}(S) = \{v \in V_m : v|_S = 0\}.$$

Always $\mathrm{rad}(A) \subseteq J_{\mathrm{act}}(Z_{\mathrm{act}}(A))$, because a power vanishing at a complex-valued evaluation forces its base to vanish there. The same definitions apply to the algebra V on all finite-component links, with each invariant having a finite order on that whole domain.

Theorem 1. *In V_m, put*

$$g(L) = \sum_{i<j} \mathrm{lk}(L_i, L_j)^2, \qquad f = g(4g-1), \qquad A = fV_m.$$

Then $Z_{\mathrm{act}}(A)$ is the nonempty set of algebraically split links, and

$$g \in J_{\mathrm{act}}(Z_{\mathrm{act}}(A)) \setminus \mathrm{rad}(A).$$

Thus $J_{\mathrm{act}}(Z_{\mathrm{act}}(A)) \neq \mathrm{rad}(A)$. The same construction gives a counterexample in V.

2 Polynomial restriction on the whole algebra

The polynomial behavior of finite-type invariants on twist or braid sequences is established prior work; see, for example, Stanford–Trapp [2, Theorem E], Eisermann [3], and Stoimenow [4, Theorem 2.1, pp. 690–691]. Stoimenow explicitly credits his earlier braiding-sequence work. We give the particular difference argument needed here, including split components, so that no extension of a character from a smaller algebra is assumed.

Let σ be the positive generator of the two-strand braid group. For integers $n \geq 0$, let L_n be the coherently oriented closure of σ^{2n}, together with $m-2$ split unknots. Its two braid components have linking number n, all other pairwise linking numbers vanish, and consequently $g(L_n) = n^2$.

Lemma 1. *For every $v \in V_m$ of order at most d, there is a unique $P_v(t) \in \mathbb{C}[t]$, of degree at most d, satisfying $P_v(n) = v(L_n)$ for all $n \geq 0$. The map $v \mapsto P_v$ is a unital complex-algebra homomorphism.*

Proof. Write $a(n) = v(L_n)$ and $\Delta a(n) = a(n+1) - a(n)$. Changing one crossing in an added block σ^2 gives $\sigma\sigma^{-1} = 1$, by a Reidemeister II cancellation. Doing this in r disjoint blocks realizes

$$\Delta^r a(n) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} v(L_{n+j})$$

as the alternating resolution of an r-singular link. Equivalently, the braid expression for that resolution is

$$\sigma^{2n}(\sigma^2 - 1)^r = \sigma^{2n+r}(\sigma - \sigma^{-1})^r.$$

Each factor $\sigma - \sigma^{-1}$ resolves one designated double point. Every resolution has an even braid exponent and closes to two components; the additional split components stay fixed. Finite type therefore gives $\Delta^{d+1} a(n) = 0$.

The usual difference recurrence yields the Newton polynomial

$$P_v(t) = \sum_{r=0}^{d} \binom{t}{r} \Delta^r a(0), \qquad \binom{t}{r} = \frac{t(t-1)\cdots(t-r+1)}{r!},$$

which agrees with $a(n)$ at all nonnegative integers. Its uniqueness follows because a nonzero complex polynomial cannot vanish at infinitely many distinct integers. The restrictions of $u+v$, uv, and 1 agree there with $P_u + P_v$, $P_u P_v$, and 1, respectively. Polynomial uniqueness proves the asserted homomorphism property. □

3 The counterexample

Proof of Theorem 1. Pairwise linking numbers are integer-valued and have order at most one. On any prescribed crossing-resolution cube, each is affine in the binary resolution variables: an intercomponent crossing change changes the relevant linking number by one up to sign, and a self-crossing change does not change it. Its square has degree at most two on that cube. Hence g has order at most two and $f = 4g^2 - g$ has order at most four. The sum defining g is unchanged by relabeling components or reversing their orientations.

For every actual link, $g(L)$ is a nonnegative integer. Thus

$$f(L) = 0 \quad \Longleftrightarrow \quad g(L) = 0 \quad \Longleftrightarrow \quad \mathrm{lk}(L_i, L_j) = 0 \text{ for all } i < j.$$

Since f generates A, this is precisely its actual-link zero set. It contains the unlink, and g vanishes on it.

By Lemma 1,

$$\chi : V_m \longrightarrow \mathbb{C}, \qquad \chi(v) = P_v(1/2)$$

is a character on the entire finite-order algebra. As $P_g(t) = t^2$, we have

$$\chi(g) = \frac{1}{4}, \qquad \chi(f) = 0.$$

If $g^N = fh$ for some $N \geq 1$ and $h \in V_m$, applying χ would give $4^{-N} = 0$, a contradiction. Therefore $g \notin \mathrm{rad}(A)$.

For V, restrict every invariant to the same fixed-component family. The same character proof applies; the linking-square invariant and its order bound are defined uniformly on all finite-component links. □

The character is not evaluation at an actual link, since its g-value is $1/4$. In particular, formal evaluation at $n = 1/2$ is not evaluation on the one-component closure of σ. The obstruction concerns the use of actual links as the evaluation set, and does not contradict the classical Hilbert Nullstellensatz.

4 Scope, attribution, and verification

The same construction and difference proof apply to ordered, labelled links as well. The source's separate ideal generated by the indexed pairwise linking numbers is understood in that labelled theory. There it differs from our principal ideal although their actual-link zero sets coincide; its radical/vanishing-ideal question is not decided here. The symmetric witness and the proof also apply after labels are forgotten, as in Theorem 1. No conclusion about completed invariant algebras, homogeneous-ideal variants, or positive characteristic is claimed. The proof also works over any characteristic-zero field and, by pulling back invariants to oriented links, after forgetting orientations.

The contribution recorded here is the explicit principal ideal and its nonempty actual-link zero-set obstruction, as an elementary application of older polynomiality. No new polynomiality method is claimed. A bounded primary-source priority audit located no explicit equivalent earlier counterexample in the bodies examined; it does not establish historical firstness. The accompanying audit states its source/version access limits and records other older routes, including the connected-sum polynomiality of Jeong–Park [5].

The accompanying exact-rational verification checks braid resolutions, Newton identities, linking-square symmetries, and the character calculation. These finite controls support the written proof; they do not replace the argument for every finite-order invariant.

AI and review disclosure. AI tools were used extensively in developing, drafting, source auditing, and adversarially checking this result and its verification materials. The work has not undergone conventional human peer review or refereeing at the time of this preprint.

References

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